Finite and Boundary Element Methods in Acoustics

W. Kreuzer, Z. Chen, H. Waubke

Austrian Academy of Sciences, Acoustics Research Institute

ARI meets NuHAG
- Finite Elements
  - Vibrations in stoch. layers
- Boundary Elements
  - Noise Barriers
  - Vibrations in Tunnels
- FMM-BEM
  - Calc. of HRTFs
Example: Laplace Equation, weighted residual

\[ \nabla^2 u = 0, \quad \int_{\Omega} \nabla^2 u \omega d\Omega = 0 \]

Gauss-Green theorem

\[ \int_{\Omega} \nabla^2 u \omega dx = \int_{\Gamma} \frac{\partial u}{\partial n} \omega d\Gamma - \int_{\Omega} \nabla u \nabla \omega d\Omega \]

Discretize \( \Omega \) with a grid of simple geometric elements and approximation of \( u \) with basis \( u(x) = \sum u_i \psi_i(x) \)

Choose weighting function \( \omega \), f.e. Galerkin: \( \psi_m(x) \)

Linear system of equations \( Ku = f \)
• Lot of possibilities to choose $\omega$
• Fundamental solution: Solution of $\nabla^2 \omega = -\delta(\xi - x)$
• Second time Gauss-Green theorem

\[
\int_{\Omega} \nabla^2 u \omega \, dx = \int_{\Gamma} \frac{\partial u}{\partial n} \omega d\Gamma - \int_{\Omega} \nabla u \nabla \omega \, d\Omega \\
= \int_{\Gamma} \frac{\partial u}{\partial n} \omega d\Gamma - \int_{\Gamma} \frac{\partial w}{\partial n} d\Gamma + \int_{\Omega} u \nabla^2 \omega d\Omega
\]

\[
\int_{\Omega} u \nabla^2 \omega \, d\Omega = -\int_{\Omega} u \delta(\xi - x) \, d\Omega = \begin{cases} 
- u(\xi) & \xi \in \Omega \\
- \frac{1}{2} u(\xi) & \xi \in \Gamma \\
0 & \xi \notin \Omega
\end{cases}
\]
\[ \kappa u(\xi) + \int_{\Gamma} u \frac{\partial \omega}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} \omega d\Gamma \]

- 2D: \( \omega = -\frac{1}{2\pi} \log r, r = ||\xi - x|| \)
- 3D: \( \omega = \frac{1}{4\pi r} \)
- Discretization \( \rightarrow \) linear system of equations
- Only necessary for points on boundary
- Once values for boundary are calculated, results for \( \xi \notin \Gamma \) are easy to get
### FEM vs BEM

<table>
<thead>
<tr>
<th>FEM</th>
<th>BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(large) sparse sym. matrix mesh for entire domain “simple” integrals widely applicable</td>
<td>(smaller) nonsym. fully pop. matrix mesh only for boundary singular integrals “restricted” to some problems</td>
</tr>
</tbody>
</table>

- What if there is no fundamental solution?
- What if the system gets too big → FMM
No fundamental solution?

- “No BEM without fundamental solution $G(\xi, x)$”
- “Solution of the problem $Lu = 0$ with a singularity at $\xi$”
- Fourier transformation $\mathcal{F}$

$$\mathcal{L}G = \delta \xrightarrow{\mathcal{F}} \hat{\mathcal{L}}\hat{G} = 1$$

- Calculation of approximation for $G$ in the Fourier domain on some grid
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel

\[ \kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma \]
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel

$$\kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma$$
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel

$$\kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma$$
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $\gamma$, results from above are taken for BEM-formulation of the tunnel

$$\kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma$$
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel

\[ \kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma \]
Vibrations in tunnels immersed in orthotropic layered soil

- Propagation of waves in soil without tunnel with pointload ($\delta$ functional) at different depths $z$
- Deformation and stresses at different depths $z$
- After Fourier backtransformation w.r.t. $y$, results from above are taken for BEM-formulation of the tunnel

\[ \kappa u(\xi) + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} G d\Gamma \]
Orthotropic layers

- No singularity in the Fourier domain
- Problems with backtransformation
- No FFT possible
- Interpolation with $\alpha e^{\beta |y|}, \alpha e^{\beta y^2}$
No singularity in the Fourier domain
Problems with backtransformation
No FFT possible
Interpolation with $\alpha e^{\beta \vert y \vert}$, $\alpha e^{\beta y^2}$
Orthotropic layers

- No singularity in the Fourier domain
- Problems with backtransformation
- No FFT possible
- Interpolation with $\alpha e^{\beta |y|}, \alpha e^{\beta y^2}$

Kreuzer, Chen, Waubke (ARI)
Localization of sound sources dependent on the form of the pinna
Calculation of acoustic pressure on the head
Model has about 30,000 nodes and over 65,000 elements
Too big for BEM → Fast Multipole Method
- Localization of sound sources dependent on the form of the pinna
- Calculation of acoustic pressure on the head
- Model has about 30,000 nodes and over 65,000 elements
- Too big for BEM → Fast Multipole Method
• Originally developed for N-body problems
• Man-in-the-middle principle
• Near field → classical BEM
• Far field → fast multipole method
• Single or multilevel
Originally developed for N-body problems
- Man-in-the-middle principle
- Near field $\rightarrow$ classical BEM
- Far field $\rightarrow$ fast multipole method
- Single or multilevel
Originally developed for N-body problems
- Man-in-the-middle principle
- Near field → classical BEM
- Far field → fast multipole method
- Single or multilevel
- Originally developed for N-body problems
- Man-in-the-middle principle
- Near field → classical BEM
- Far field → fast multipole method
- Single or multilevel
Originally developed for N-body problems
- Man-in-the-middle principle
- Near field $\rightarrow$ classical BEM
- Far field $\rightarrow$ fast multipole method
- Single or multilevel
Helmholtz and FMM

- Helmholtz equation

\[ \Delta^2 \Phi(x) + k^2 \Phi(x) = 0 \]

- Fundamental solution:

\[ G(x, y) = \frac{e^{ikr}}{4\pi r}, \quad r = ||x - y|| \]

- Expansion of \( G(x,y) \) possible

\[
\frac{e^{i|D+d|}}{|D + d|} = \frac{ik}{4\pi} \sum_{\ell} (2\ell + 1)i^\ell h_\ell(kD) \int_S e^{iskd} P_\ell(s\hat{D}) ds
\]

with \( \hat{D} = \frac{D}{||D||} \), \( h_\ell(x) \) Hankel functions, \( P_\ell(x) \) Legendre polynomials (\( h_\ell \to \infty \) for \( \ell \to \infty \))
Helmholtz and FMM

- Helmholtz equation

\[ \Delta^2 \Phi(x) + k \Phi(x) = 0 \]

- Fundamental solution:

\[ G(x, y) = \frac{e^{ikr}}{4\pi r}, \quad r = ||x - y|| \]

- Expansion of \( G(x,y) \) possible

\[ \Phi(x) = \frac{ik}{4\pi} \int_{S} e^{ik(x-z_2)s} M_L(s, z_2 - z_1) \sum_{a=1}^{A} e^{ik(z_1 - y_a)s} q_a ds \]
Acknowledgments/Literature

- Vibrations in orthotropic layers: BMVIT/FFG Pr. 809089
- HRTFs: FWF Pr. P-18401B15
- Peter Hunter: FEM/BEM Notes
- Matthias Fischer: The Fast Multipole Boundary Element Method and its Application to Structured-Acoustic Field Interaction
- Z. Chen et al: A Formulation of the Fast Multipole Boundary Element Method (FMBEM) for Acoustic Radiation and Scattering from Three-Dimensional Structures, to appear