Almost Lossless Analog Compression without Phase Information

Erwin Riegler
ETH Zurich, 8092 Zurich, Switzerland
eriegler@nari.ee.ethz.ch

Georg Tauböck
Austrian Academy of Sciences, 1040 Vienna, Austria
georg.tauboeck@oeaw.ac.at

***Abstract*—We propose an information-theoretic framework for phase retrieval. Specifically, we consider the problem of recovering an unknown vector $\mathrm{x} \in \mathbb{R}^n$ up to an overall sign factor from $m = \lfloor Rn \rfloor$ phaseless measurements with compression rate R and derive a general achievability bound for R. Surprisingly, it turns out that this bound on the compression rate is the same as the one for almost lossless analog compression obtained by Wu and Verdú (2010): Phaseless linear measurements are "as good" as linear measurements with full phase information in the sense that ignoring the sign of m measurements only leaves us with an ambiguity with respect to an overall sign factor of x.**

I. Introduction

In many different areas of science, physical limitations make it impossible to measure the sign (phase in the complex case) of a signal but obtaining amplitudes is relatively easy. Well known examples are X-ray crystallography, astronomy, or diffraction imaging [1]–[3]. The problem of retrieving a signal up to a global sign (phase in the complex case) from intensity measurements is often referred to as *phase retrieval*. More formally, let $\mathbb{R}^n_{\sim}$ be the set of equivalence classes $[\mathrm{x}] = \{\mathrm{x}\} \cup \{-\mathrm{x}\}$ with $\mathrm{x} \in \mathbb{R}^n$. Phase retrieval is the problem of recovering $[\mathrm{x}] \in \mathbb{R}^n_{\sim}$ from m phaseless measurements of the form[1] $\mathrm{y} = |\mathrm{A}\mathrm{x}| \in \mathbb{R}^m$ with measurement matrix $\mathrm{A} \in \mathbb{R}^{m \times n}$.

It is by no means clear how large m has to be to allow for recovery of $[\mathrm{x}] \in \mathbb{R}^n_{\sim}$ from m phaseless measurements. Thus from the very beginning, there have been a number of works regarding recovery conditions for this problem in the context of specific applications [4]. More recently, this question has been studied in more abstract terms, asking for the smallest number m of phaseless measurements that is required to make the mapping $[\mathrm{x}] \mapsto |\mathrm{A}\mathrm{x}|$ injective without imposing structural assumptions on A. In [5], the authors showed that at least $2n - 1$ such measurements are necessary and generically sufficient to guarantee injectivity. Furthermore, it was shown that semidefinite programming can be used to recover $[\mathrm{x}]$ if $\mathbf{A}$ is random with i.i.d. Gaussian entries or with i.i.d. rows that are uniformly distributed on a sphere, as long as $m \geq c_0 n$ for a sufficiently large constant c_0 [6]. Other phase retrieval methods for which theoretical performance guarantees are available can be found, e.g., in [7]–[10].

Recently, there has been also interest in *sparse phase retrieval*, where the number s of nonzero coefficients of the vector x is much smaller than n. This a-priori knowledge about x can be used to reduce the number of measurements significantly. For instance, $\mathcal{O}(s \log(n/s))$ measurements were shown to be sufficient for stable sparse phase retrieval [11]. If the rows of the measurement matrix A are a generic choice of vectors in $\mathbb{R}^n$, injectivity of the mapping $[\mathrm{x}] \mapsto |\mathrm{A}\mathrm{x}|$ is guaranteed provided that $m \geq 2s$ [12].

Contributions: Following the approach introduced for compressed sensing [13] and signal separation [14] problems, we formulate phase retrieval as an analog source coding problem. Assuming that the unknown vector $\mathbf{x}$ is random with a certain distribution, we derive asymptotic recovery results for $[\mathbf{x}]$. Our results hold for Lebesgue almost all (a.a.) measurement matrices A. However, our results are in terms of probability of error (with respect to the distribution of $\mathbf{x}$) and hence do not provide worst-case guarantees. Specifically, we study the asymptotic setting $n \to \infty$ where the vector x is a realization of a random process; for each n, we let $m = \lfloor Rn \rfloor$ for a parameter R, which we denote *compression rate*. In Theorem 1 we show that we can recover $[\mathbf{x}]$ from m phaseless measurements with arbitrarily small probability of error for a.a. measurement matrices A, provided that n is sufficiently large and the compression rate R is larger than the (lower) Minkowski dimension compression rate (see Definition 4) of $\mathbf{x}$. It is remarkable that the obtained result is identical to the corresponding result in compressive sensing [13] where $\mathrm{y} = \mathrm{A}\mathrm{x}$, so that we can conclude that *in terms of achievability results, phaseless linear measurements are "as good" as linear measurements with full phase information:* Ignoring the sign of m measurements only leaves us with an ambiguity with respect to an overall sign factor of x.

Notation: Roman letters $\mathrm{A}, \mathrm{B}, \dots$ and $\mathrm{a}, \mathrm{b}, \dots$ designate deterministic matrices and vectors, respectively. Boldface letters $\mathbf{A}, \mathbf{B}, \dots$ and $\mathbf{a}, \mathbf{b}, \dots$ denote random matrices and random vectors, respectively. For the distribution of a random matrix $\mathbf{A}$ and a random vector $\mathbf{a}$, we write $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{a}}$, respectively. The ith component of the vector u (random vector $\mathbf{u}$) is u_i ($\mathbf{u}_i$). The superscript $^\mathsf{T}$ stands for transposition. For a matrix A, $\mathrm{tr}(\mathrm{A})$ denotes its trace. The identity matrix of suitable size is denoted by I. For a vector u, we write $\|\mathrm{u}\| = \sqrt{\mathrm{u}^\mathsf{T}\mathrm{u}}$ for its Euclidean norm. For the Euclidean space $(\mathbb{R}^k, \|\cdot\|)$, we denote the open ball of radius r centered at $\mathrm{u} \in \mathbb{R}^k$ by $\mathcal{B}_k(\mathrm{u}, r)$, $V(k, r)$ stands for its volume. The Borel sigma algebra on $\mathbb{R}$ is denoted by $\mathscr{B}_{\mathbb{R}}$. We write $\mathbb{R}_{\geq}$ for the set of nonnegative real numbers with Borel sigma algebra $\mathscr{B}_{\mathbb{R}_{\geq}}$. For $\mathrm{u}, \mathrm{v} \in \mathbb{R}^k$,

This work was supported in part by the WWTF under grant VRG 12-009 and by the FWF under grant Y 551-N13.

[1]For a vector $\mathrm{u} \in \mathbb{R}^k$, we define the element-wise absolute value operation as $|\mathrm{u}| = (|u_1|, \dots, |u_k|)^\mathsf{T}$.

$u \sim v$ means that either $u = v$ or $u = -v$ and we write for the corresponding equivalence classes $[u] = \{u\} \cup \{-u\}$. For a set $\mathcal{S} \subseteq \mathbb{R}^k$, $\mathcal{S}_\sim = \{[u] \mid u \in \mathcal{S}\}$. The indicator function on a set $\mathcal{U}$ is denoted by $\chi_\mathcal{U}$.

II. Main Results

We start by formulating phase retrieval as a source coding problem.

Definition 1. *(Source vector) Let $(\mathsf{x}_i)_{i\in\mathbb{N}}$ be a stochastic process on $(\mathbb{R}^\mathbb{N}, \mathscr{B}_\mathbb{R}^{\otimes\mathbb{N}})$. Then, for $n \in \mathbb{N}$, the source vector $\mathbf{x}$ of length n is given by $\mathbf{x} = (\mathsf{x}_1,\dots,\mathsf{x}_n)^\mathsf{T} \in \mathbb{R}^n$.*

Definition 2. *(Code, achievable rate) For $\mathbf{x}$ as in Definition 1 and $\varepsilon > 0$, an (n,m) code consists of*

(i) measurements $|\mathrm{A}\cdot| : \mathbb{R}^n \to \mathbb{R}^m_\geq$;

(ii) a decoder $g : \mathbb{R}^m_\geq \to \mathbb{R}^n$ that is measurable with respect to $\mathscr{B}^{\otimes m}_{\mathbb{R}_\geq}$ and $\mathscr{B}^{\otimes n}_\mathbb{R}$.

We call R with $0 < R \leq 1$ an ε-achievable rate if there exists an $N(\varepsilon) \in \mathbb{N}$ and a sequence of $(n, \lfloor Rn\rfloor)$ codes with decoders g such that

$$\mathrm{P}[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}] \leq \varepsilon$$

for all $n \geq N(\varepsilon)$.

Next, we introduce the Minkowski dimension compression rate for source vectors.

Definition 3. *(Minkowski dimension) Let $\mathcal{U}$ be a nonempty bounded set in $\mathbb{R}^n$. The lower Minkowski dimension of $\mathcal{U}$ is defined as*

$$\underline{\dim}_\mathrm{B}(\mathcal{U}) = \liminf_{\rho\to 0} \frac{\log N_\mathcal{U}(\rho)}{\log\frac{1}{\rho}}$$

and the upper Minkowski dimension of $\mathcal{U}$ is defined as

$$\overline{\dim}_\mathrm{B}(\mathcal{U}) = \limsup_{\rho\to 0} \frac{\log N_\mathcal{U}(\rho)}{\log\frac{1}{\rho}}$$

where $N_\mathcal{U}(\rho)$ is the covering number of $\mathcal{U}$ given by

$$N_\mathcal{U}(\rho) = \min\Big\{k \in \mathbb{N} \mid \mathcal{U} \subseteq \bigcup_{i\in\{1,\dots,k\}} \mathcal{B}_n(u_i,\rho),\ u_i \in \mathbb{R}^n\Big\}.$$

If $\underline{\dim}_\mathrm{B}(\mathcal{U}) = \overline{\dim}_\mathrm{B}(\mathcal{U})$, we write $\dim_\mathrm{B}(\mathcal{U})$.

Definition 4. *(Minkowski dimension compression rate) For $\mathbf{x}$ from Definition 1 and $\varepsilon > 0$, we define the lower Minkowski dimension compression rate as*

$$\underline{R}_\mathrm{B}(\varepsilon) = \limsup_{n\to\infty} \underline{a}_n(\varepsilon), \quad \text{where}$$
$$\underline{a}_n(\varepsilon) = \inf\Big\{\frac{\underline{\dim}_\mathrm{B}(\mathcal{U})}{n} \;\Big|\; \mathcal{U} \subset \mathbb{R}^n,\ \mathrm{P}[\mathbf{x}\in\mathcal{U}] \geq 1-\varepsilon\Big\}.$$

and the upper Minkowski dimension compression rate as

$$\overline{R}_\mathrm{B}(\varepsilon) = \limsup_{n\to\infty} \overline{a}_n(\varepsilon), \quad \text{where}$$
$$\overline{a}_n(\varepsilon) = \inf\Big\{\frac{\overline{\dim}_\mathrm{B}(\mathcal{U})}{n} \;\Big|\; \mathcal{U} \subset \mathbb{R}^n,\ \mathrm{P}[\mathbf{x}\in\mathcal{U}] \geq 1-\varepsilon\Big\}.$$

The sets $\mathcal{U}$ in the definitions for $\underline{a}_n(\varepsilon)$ and $\overline{a}_n(\varepsilon)$ are assumed to be nonempty and bounded.

Example 1. The source vector $\mathbf{x}$ from Definition 1 has a mixed discrete-continuous distribution if for each $n \in \mathbb{N}$ the random variables x_i, $i \in \{1,\dots,n\}$, are independent and distributed according to

$$\mu_{\mathsf{x}_i} = (1-\lambda)\mu_d + \lambda\mu_c, \quad i \in \{1,\dots,n\}$$

where $0 \leq \lambda \leq 1$ is the mixing parameter, μ_c is a distribution on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, absolutely continuous with respect to Lebesgue measure, and μ_d is a discrete distribution. Then, [13, Th. 15]

$$\underline{R}_\mathrm{B}(\varepsilon) = \overline{R}_\mathrm{B}(\varepsilon) = \lambda, \quad 0 < \varepsilon < 1.$$

The following result states that every rate $R > \underline{R}_\mathrm{B}(\varepsilon)$ is ε-achievable for Lebesgue a.a. matrices A.

Theorem 1. *Let $0 < \varepsilon < 1$ and $\mathbf{x}$ as in Definition 1. Then, for Lebesgue a.a. matrices $\mathrm{A} \in \mathbb{R}^{m\times n}$ with $m = \lfloor Rn\rfloor$, R is an ε-achievable rate provided that $R > \underline{R}_\mathrm{B}(\varepsilon)$.*

Proof. Since $R > \underline{R}_\mathrm{B}(\varepsilon)$ and $m = \lfloor Rn\rfloor$, Definition 4 implies that there exists a sequence of nonempty bounded sets $\mathcal{U}_n \subseteq \mathbb{R}^n$ and an $N(\varepsilon) \in \mathbb{N}$ such that

$$\underline{\dim}_\mathrm{B}(\mathcal{U}) < m \quad (1)$$
$$\mathrm{P}\big[\mathbf{x}\in\mathcal{U}\big] \geq 1-\varepsilon \quad (2)$$

for all $\mathcal{U} = \mathcal{U}_n$ with $n \geq N(\varepsilon)$. In the remainder of the proof we assume that n is sufficiently large for (1) and (2) to hold. The claim now follows from Proposition 1 below. □

Proposition 1. *Let $\varepsilon \geq 0$, $\mathbf{x} \in \mathbb{R}^n$ a random vector, and $\mathcal{U} \subseteq \mathbb{R}^n$ a nonempty bounded set with $\mathrm{P}[\mathbf{x}\in\mathcal{U}] \geq 1-\varepsilon$. Then, for Lebesgue a.a. matrices $\mathrm{A} \in \mathbb{R}^{m\times n}$, there exists a decoder g with $\mathrm{P}[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}] \leq \varepsilon$ provided that $\underline{\dim}_\mathrm{B}(\mathcal{U}) < m$.*

Proof. See Section III. □

Remark 1. By [15, Sec. 3.2, Properties (i)–(iii)], the lower Minkowski dimension of any bounded nonempty subset in $\mathbb{R}^n$ containing only vectors with no more than s nonzero entries is at most s. Therefore, Proposition 1 implies that any s-sparse random vector $\mathbf{x} \in \mathbb{R}^n$ can be recovered with arbitrarily small probability of error (by increasing the size of the set $\mathcal{U}$ in Proposition 1), provided that $m > s$. This result holds for an arbitrary distribution of $\mathbf{x}$ and a.a. matrices $\mathrm{A} \in \mathbb{R}^{m\times n}$. The best known recovery threshold for deterministic s-sparse vectors is $m \geq 2s$ [12].

Remark 2. It is worth noting that formally phase retrieval can be formulated as a matrix completion problem with measurements $y_i^2 = \mathrm{tr}(a_i a_i^\mathsf{T} x x^\mathsf{T})$ using rank-one measurement matrices $\mathrm{A}_i = a_i a_i^\mathsf{T}$, $i = 1,\dots,m$. However, compared to the rank-one measurement matrices used in the matrix completion problem [16], [17], the matrices $a_i a_i^\mathsf{T}$ are symmetric. This complicates the proof of Proposition 1 significantly and forces us to develop a novel concentration of measure result (Lemma 3). On the other hand, in phase retrieval we are interested in recovering symmetric rank-one matrices $x x^\mathsf{T}$ (which is equivalent to the recovery of $[\mathrm{x}]$), whereas matrix completion deals with the recovery of arbitrary low-rank matrices.

In the mixed discrete-continuous case we can strengthen the result of Theorem 1 through the following lemma.

Lemma 1. *Let $0 < \varepsilon < 1$ and $\mathbf{x}$ be distributed according to the mixed discrete-continuous distribution in Example 1 with mixing parameter λ. Then, for Lebesgue a.a. matrices $\mathrm{A} \in \mathbb{R}^{m\times n}$ with $m = \lfloor Rn \rfloor$, R is ε-achievable provided that $R > \lambda$. Moreover, $R \geq \lambda$ is also a necessary condition for R being ε-achievable.*

Proof. Achievability: Follows from Theorem 1 and Example 1. Converse: Suppose that a rate $R < \lambda$ is ε-achievable for some ε with $0 < \varepsilon < 1$. This implies that there exists a set $\mathcal{K} \subseteq \mathbb{R}^n$ and a matrix $\mathrm{A} \in \mathbb{R}^{m\times n}$ with $m = \lfloor Rn \rfloor$ such that
(a) $\Pr[\mathbf{x} \in \mathcal{K}] \geq 1 - \varepsilon$;
(b) $|\mathrm{A} \cdot |$ is one-to-one on $\mathcal{K}_\sim$
for n sufficiently large. From (b) it follows that there can be at most one equivalence class $[\mathrm{u}] \in \mathcal{K}_\sim$ with $\mathrm{Au} = \mathrm{A}(-\mathrm{u}) = 0$ (if there was more than one such equivalence class then the mapping $|\mathrm{A}\cdot|$ would not be one-to-one on $\mathcal{K}_\sim$).

Suppose first that there is no equivalence class $[\mathrm{u}] = \{\mathrm{u}, -\mathrm{u}\} \in \mathcal{K}_\sim$ with $\mathrm{Au} = \mathrm{A}(-\mathrm{u}) = 0$ and $\mathrm{u} \neq 0$. Then, (b) implies that A is one-to-one on $\mathcal{K}$ which, together with (a) and $R < \lambda$, leads to a contradiction to the converse part of [13, Thm. 6].

Now suppose that there is an equivalence class $[\mathrm{u}] = \{\mathrm{u}, -\mathrm{u}\} \in \mathcal{K}_\sim$ with $\mathrm{Au} = \mathrm{A}(-\mathrm{u}) = 0$ and $\mathrm{u} \neq 0$. Let $\tilde{R}$ be such that $R < \tilde{R} < \lambda$ and set $\tilde{m} = \lfloor \tilde{R}n \rfloor$. Then, $\tilde{m} > m$ for n sufficiently large. Let $\tilde{\mathrm{A}} = (\mathrm{A}^\mathsf{T}, \mathrm{u}, 0,..., 0)^\mathsf{T} \in \mathbb{R}^{\tilde{m}\times n}$. Then, (b) implies that $\tilde{\mathrm{A}}$ is one-to-one on $\mathcal{K}$ which, together with (a) and $\tilde{R} < \lambda$, leads to a contradiction to the converse part of [13, Thm. 6]. ☐

III. Proof of Proposition 1

Let

$$\mathcal{F}(\mathrm{y}) = \big\{\mathrm{u} \in \mathbb{R}^n \big| \mathrm{u} \in \mathcal{U}, |\mathrm{Au}| = \mathrm{y}\big\} \cup \big\{\mathrm{u} \in \mathbb{R}^n \big| -\mathrm{u} \in \mathcal{U}, |\mathrm{Au}| = \mathrm{y}\big\}, \quad \mathrm{y} \in \mathbb{R}^m_{\geq}.$$

For a vector $\mathrm{u} \in \mathcal{F}(\mathrm{y}) \setminus \{0\}$, let $\bar{u}$ denote the first nonzero component of u. We then define the reduced set

$$\bar{\mathcal{F}}(\mathrm{y}) = \big\{\mathrm{u} \in \mathcal{F}(\mathrm{y}) \setminus \{0\} \big| \bar{u} = |\bar{u}|\big\} \cup (\mathcal{F}(\mathrm{y}) \cap \{0\}), \quad \mathrm{y} \in \mathbb{R}^m_{\geq}.$$

We define the decoder $g : \mathbb{R}^m_{\geq} \to \mathbb{R}^n$ by

$$g(\mathrm{y}) = \begin{cases} \mathrm{u}, & \text{if } \bar{\mathcal{F}}(\mathrm{y}) = \{\mathrm{u}\} \\ \mathrm{e}, & \text{else} \end{cases}$$

where e is some fixed vector in the complement of $\mathcal{U}$ (used to declare a decoding error). Then, we have

$$\begin{aligned} &\mathrm{P}\big[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}\big] \\ &= \mathrm{P}\big[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}, \mathbf{x} \in \mathcal{U}\big] + \mathrm{P}\big[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}, \mathbf{x} \notin \mathcal{U}\big] \\ &\leq \mathrm{P}\big[g(|\mathrm{A}\mathbf{x}|) \not\sim \mathbf{x}, \mathbf{x} \in \mathcal{U}\big] + \varepsilon \\ &= \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \big| \mathrm{u} \not\sim \mathbf{x}, |\mathrm{Au}| = |\mathrm{A}\mathbf{x}|, \mathbf{x} \in \mathcal{U}\big] + \varepsilon \end{aligned} \tag{3}$$

where (3) follows from the definition of the decoder. Fix an arbitrary $r > 0$. Suppose that we can show that

$$P(\mathrm{x}) = \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \text{ with } \mathrm{u} \not\sim \mathrm{x}, |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big] = 0, \quad \mathrm{x} \in \mathcal{U} \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ has independent rows that are uniformly distributed on $\mathcal{B}_n(0, r)$. Then,

$$\begin{aligned} &\int_{\mathcal{A}(r)} \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \big| \mathrm{u} \not\sim \mathbf{x}, |\mathrm{Au}| = |\mathrm{A}\mathbf{x}|, \mathbf{x} \in \mathcal{U}\big] \,\mathrm{d}\mu_{\mathbf{A}} \\ &= \int_{\mathcal{U}} \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \text{ with } \mathrm{u} \not\sim \mathrm{x}, |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big] \,\mathrm{d}\mu_{\mathbf{x}} \\ &= 0 \end{aligned} \tag{5}$$

where we used Fubini's Theorem and set $\mathcal{A}(r) = \mathcal{B}_n(0, r) \times ... \times \mathcal{B}_n(0, r)$. Since r is arbitrary, (5) implies that

$$\mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \big| \mathrm{u} \not\sim \mathbf{x}, |\mathrm{Au}| = |\mathrm{A}\mathbf{x}|, \mathbf{x} \in \mathcal{U}\big] = 0 \tag{6}$$

for Lebesgue a.a. matrices A. Hence, combining (3) and (6) proves the Proposition provided that we can show that (4) holds, which is done in Section IV.

IV. Proof of (4)

Suppose first that $\mathrm{x} = 0$. Then, $P(\mathrm{x}) = 0$ if and only if

$$\mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \setminus \{0\} \text{ with } \mathbf{A}\mathrm{u} = 0\big] = 0. \tag{7}$$

Since $\underline{\dim}_\mathrm{B}(\mathcal{U}) < m$, (7) follows from [14, Prop. 1]. Therefore, we can assume in what follows that $\mathrm{x} \neq 0$.

We can upper-bound $P(\mathrm{x}) \leq P_1(\mathrm{x}) + P_2(\mathrm{x})$ with

$$P_i(\mathrm{x}) = \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U}_i(\mathrm{x}) \text{ with } |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big], \quad i \in \{1, 2\}$$

where we defined

$$\begin{aligned} \mathcal{U}_1(\mathrm{x}) &= \{\mathrm{u} \in \mathcal{U} \mid \operatorname{rank}(\mathrm{x}, \mathrm{u}) = 2\} \\ \mathcal{U}_2(\mathrm{x}) &= \{\mathrm{u} \in \mathcal{U} \mid \operatorname{rank}(\mathrm{x}, \mathrm{u}) = 1\} \setminus \{\mathrm{u} \in \mathcal{U} | \mathrm{u} \sim \mathrm{x}\}. \end{aligned}$$

We have to show that $P_i(\mathrm{x}) = 0$ for $i \in \{1, 2\}$. First, we establish $P_2(\mathrm{x}) = 0$. We have (recall that $\mathrm{x} \neq 0$)

$$\begin{aligned} &P_2(\mathrm{x}) \\ &= \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{U} \text{ with } \operatorname{rank}(\mathrm{x}, \mathrm{u}) = 1, \mathrm{u} \not\sim \mathrm{x}, |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big] \\ &= \mathrm{P}\big[\mathbf{A}\mathrm{x} = 0\big] \\ &= 0 \end{aligned}$$

where we used [14, Prop. 1] together with $\underline{\dim}_\mathrm{B}\big(\{\mathrm{x}\}\big) = 0$ in the last step. It remains to show that $P_1(\mathrm{x}) = 0$. To this end, we first present an auxiliary lemma.

Lemma 2. *Let $r > 0$, $\emptyset \neq \mathcal{S} \subseteq \mathcal{B}_n(0, L)$, $\rho > 0$, $\mathrm{x} \in \mathcal{B}_n(0, L)$, and $\mathbf{A} \in \mathbb{R}^{m\times n}$ with independent rows that are uniformly distributed on $\mathcal{B}_n(0, r)$. Then, there exist $\mathrm{s}_l(\rho) \in \mathcal{S}$, $l = 1,..., N_\mathcal{S}(\rho)$ with $N_\mathcal{S}(\rho)$ being the covering number of $\mathcal{S}$, such that*

$$\begin{aligned} &\mathrm{P}\big[\exists \mathrm{u} \in \mathcal{S} \textit{ with } \big\||\mathbf{A}\mathrm{u}| - |\mathbf{A}\mathrm{x}|\big\| \leq \rho\big] \\ &\leq \sum_{l=1}^{N_\mathcal{S}(\rho)} \mathrm{P}\big[\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)|^2 - |\mathbf{a}^\mathsf{T}\mathrm{x}|^2\big| \leq 2Lr(2r+1)\rho\big]^m \end{aligned} \tag{8}$$

where $\mathbf{a}$ *is uniformly distributed on* $\mathcal{B}_n(0,r)$.

Proof. Let $\mathcal{S} \subseteq \bigcup_{l\in\{1,\dots,N_{\mathcal{S}}(\rho)\}} \mathcal{B}_n(\mathrm{v}_l(\rho),\rho)$, $\mathrm{v}_l(\rho) \in \mathbb{R}^n$, be a *minimal* covering of $\mathcal{S}$ according to the definition of the covering number, cf. Definition 3. Then, there exist $\mathrm{s}_l(\rho) \in \mathcal{S} \cap \mathcal{B}_n(\mathrm{v}_l(\rho),\rho)$ for all $l = 1,\dots,N(\rho)$. Hence, the balls $\mathcal{B}_n(\mathrm{s}_l(\rho),2\rho)$ cover the set $\mathcal{S}$ and have centers in $\mathcal{S}$. We can upper bound the lhs in (8) by

$$\begin{aligned}
&\mathrm{P}\big[\exists \mathrm{u} \in \mathcal{S} \text{ with } \big||\mathbf{A}\mathrm{u}| - |\mathbf{A}\mathrm{x}|\big| \le \rho\big] \\
&\le \sum_{l=1}^{N_{\mathcal{S}}(\rho)} \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{S}\cap\mathcal{B}_n(\mathrm{s}_l(\rho),2\rho) \text{ with } \big||\mathbf{A}\mathrm{u}| - |\mathbf{A}\mathrm{x}|\big| \le \rho\big] \\
&\le \sum_{l=1}^{N_{\mathcal{S}}(\rho)} \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{S}\cap\mathcal{B}_n(\mathrm{s}_l(\rho),2\rho) \text{ with } \big||\mathbf{a}^\mathsf{T}\mathrm{u}| - |\mathbf{a}^\mathsf{T}\mathrm{x}|\big| \le \rho\big]^m
\end{aligned} \tag{9}$$

where (9) follows from the fact that the rows of $\mathbf{A}$ are independent and uniformly distributed on $\mathcal{B}_n(0,r)$. Using the triangle inequality we obtain

$$\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| - |\mathbf{a}^\mathsf{T}\mathrm{x}|\big| \le \big||\mathbf{a}^\mathsf{T}\mathrm{x}| - |\mathbf{a}^\mathsf{T}\mathrm{u}|\big| + \big||\mathbf{a}^\mathsf{T}\mathrm{u}| - |\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)|\big|. \tag{10}$$

The second term on the rhs of (10) can be further upper bounded by

$$\begin{aligned}
\big||\mathbf{a}^\mathsf{T}\mathrm{u}| - |\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)|\big| &\le \big|\mathbf{a}^\mathsf{T}(\mathrm{u} - \mathrm{s}_l(\rho))\big| \\
&\le \|\mathbf{a}\|\|\mathrm{u} - \mathrm{s}_l(\rho)\| \\
&\le 2r\rho
\end{aligned} \tag{11}$$

where (11) follows from $\mathrm{u} \in \mathcal{B}_n(\mathrm{s}_l(\rho),2\rho)$ and $\mathbf{a} \in \mathcal{B}_n(0,r)$. Combining (10) and (11) gives

$$\big||\mathbf{a}^\mathsf{T}\mathrm{x}| - |\mathbf{a}^\mathsf{T}\mathrm{u}|\big| \ge \big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| - |\mathbf{a}^\mathsf{T}\mathrm{x}|\big| - 2r\rho. \tag{12}$$

Using (12) in (9) yields

$$\begin{aligned}
&\mathrm{P}\big[\exists \mathrm{u} \in \mathcal{S} \text{ with } \big||\mathbf{A}\mathrm{u}| - |\mathbf{A}\mathrm{x}|\big| \le \rho\big] \\
&\le \sum_{l=1}^{N_{\mathcal{S}}(\rho)} \mathrm{P}\big[\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| - |\mathbf{a}^\mathsf{T}\mathrm{x}|\big| \le (2r+1)\rho\big]^m \\
&\le \sum_{l=1}^{N_{\mathcal{S}}(\rho)} \mathrm{P}\big[\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)|^2 - |\mathbf{a}^\mathsf{T}\mathrm{x}|^2\big| \le 2Lr(2r+1)\rho\big]^m
\end{aligned} \tag{13}$$

where (13) follows from $\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)|^2 - |\mathbf{a}^\mathsf{T}\mathrm{x}|^2\big| = \big|(|\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| + |\mathbf{a}^\mathsf{T}\mathrm{x}|)(|\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| - |\mathbf{a}^\mathsf{T}\mathrm{x}|)\big| \le 2Lr\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l(\rho)| - |\mathbf{a}^\mathsf{T}\mathrm{x}|\big|$. □

We now continue with the proof of $P_1(\mathrm{x}) = 0$. Since $\mathcal{U}$ is a bounded set, there exists an $L \in \mathbb{R}$ such that

$$\|\mathrm{u}\| \le L, \quad \mathrm{u} \in \mathcal{U}. \tag{14}$$

We define the sets $\mathcal{T}_j(\mathrm{x})$ by

$$\mathcal{T}_j(\mathrm{x}) = \left\{\mathrm{u} \in \mathcal{U}_1(\mathrm{x}) \,\middle|\, \sqrt{\|\mathrm{u}\|^2\|\mathrm{x}\|^2 - |\mathrm{u}^\mathsf{T}\mathrm{x}|^2} > \frac{1}{j}\right\}, \quad j \in \mathbb{N}.$$

Since

$$P_1(\mathrm{x}) \le \sum_{j\in\mathbb{N}} \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{T}_j(\mathrm{x}) \text{ with } |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big]$$

it is sufficient to prove that

$$P_1^{(j)}(\mathrm{x}) = \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{T}_j(\mathrm{x}) \text{ with } |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big] = 0$$

for all $j \in \mathbb{N}$. Suppose, by contradiction, that there exists a $j \in \mathbb{N}$ such that $P_1^{(j)}(\mathrm{x}) > 0$. Then,

$$\liminf_{\rho\to 0} \frac{\log P_1^{(j)}(\mathrm{x})}{\log \frac{1}{\rho}} = 0. \tag{15}$$

Furthermore, $\mathcal{T}_j(\mathrm{x}) \neq \emptyset$ and by [15, Sec. 3.2, Property (ii)] (recall that $\mathcal{T}_j(\mathrm{x}) \subseteq \mathcal{U}_1(\mathrm{x}) \subseteq \mathcal{U}$) we get

$$\underline{\dim}_\mathrm{B}\big(\mathcal{T}_j(\mathrm{x})\big) < m. \tag{16}$$

We have

$$\begin{aligned}
&\liminf_{\rho\to 0} \frac{\log P_1^{(j)}(\mathrm{x})}{\log \frac{1}{\rho}} \\
&= \liminf_{\rho\to 0} \frac{\log \mathrm{P}\big[\exists \mathrm{u} \in \mathcal{T}_j(\mathrm{x}) \text{ with } |\mathbf{A}\mathrm{u}| = |\mathbf{A}\mathrm{x}|\big]}{\log \frac{1}{\rho}} \\
&\le \liminf_{\rho\to 0} \\
&\quad \frac{\log\Big(\sum_{l=1}^{N_{\mathcal{T}_j(\mathrm{x})}(\rho)} \mathrm{P}\Big[\big||\mathbf{a}^\mathsf{T}\mathrm{s}_l^{(j)}(\rho,\mathrm{x})|^2 - |\mathbf{a}^\mathsf{T}\mathrm{x}|^2\big| \le \tilde{\rho}\Big]^m\Big)}{\log \frac{1}{\rho}} && (17) \\
&\le \liminf_{\rho\to 0} \frac{\log\Big(\tilde{\rho}^m \sum_{l=1}^{N_{\mathcal{T}_j(\mathrm{x})}(\rho)} f\big(\tilde{\rho}, r, \mathrm{s}_l^{(j)}(\rho,\mathrm{x}), \mathrm{x}\big)^m\Big)}{\log \frac{1}{\rho}} && (18) \\
&\le \liminf_{\rho\to 0} \frac{\log\big(\tilde{\rho}^m N_{\mathcal{T}_j(\mathrm{x})}(\rho) \tilde{f}(\tilde{\rho}, r, L, j)^m\big)}{\log \frac{1}{\rho}} && (19) \\
&= \underline{\dim}_\mathrm{B}(\mathcal{T}_j(\mathrm{x})) - m + m \lim_{\rho\to 0} \frac{\log \tilde{f}(\tilde{\rho}, r, L, j)}{\log \frac{1}{\rho}} \\
&= \underline{\dim}_\mathrm{B}(\mathcal{T}_j(\mathrm{x})) - m \\
&< 0 && (20)
\end{aligned}$$

where in (17) we applied Lemma 2 with $\mathcal{S} = \mathcal{T}_j(\mathrm{x})$ and set $\tilde{\rho} = 2Lr(2r+1)\rho$, (18) follows from Lemma 3 below with $\mathrm{u} = \mathrm{s}_l^{(j)}(\rho,\mathrm{x})$, $\mathrm{v} = \mathrm{x}$, and $\delta = \tilde{\rho}$ where f is defined in (22), in (19) we used that

$$\begin{aligned}
&f\big(\tilde{\rho}, r, \mathrm{s}_l^{(j)}(\rho,\mathrm{x}), \mathrm{x}\big) \\
&\le \tilde{f}(\tilde{\rho}, r, L, j) \\
&= \frac{2(2r)^{n-2}j}{V(n,r)}\Big(1 + \log\Big(2 + \frac{8r^2L^2}{\tilde{\rho}}\Big)\Big), \quad l = 1,\dots,N_{\mathcal{T}_j(\mathrm{x})}(\rho)
\end{aligned}$$

which follows from (14) and the fact that $\mathrm{s}_l^{(j)}(\rho,\mathrm{x}) \in \mathcal{T}_j(\mathrm{x})$, $l = 1,\dots,N_{\mathcal{T}_j(\mathrm{x})}(\rho)$, and in (20) we applied (16). But (20) is a contradiction to (15). Therefore, $P_1^{(j)}(\mathrm{x}) = 0$ for all $j \in \mathbb{N}$, which implies in turn that $P_1(\mathrm{x}) = 0$ and concludes the proof of (4).

V. Concentration of Measure Result

Lemma 3. *Let* $r > 0$, $\mathbf{a}$ *be uniformly distributed on* $\mathcal{B}_n(0,r)$, $\mathrm{C} = \mathrm{u}\mathrm{u}^\mathsf{T} - \mathrm{v}\mathrm{v}^\mathsf{T}$ *with linearly independent vectors* $\mathrm{u}, \mathrm{v} \in \mathbb{R}^n$, *and* $\delta > 0$. *Then*

$$\mathrm{P}\big[|\mathbf{a}^\mathsf{T}\mathrm{C}\mathbf{a}| \le \delta\big] \le \delta f(\delta, r, \mathrm{u}, \mathrm{v}) \tag{21}$$

with

$$f(\delta, r, \mathrm{u}, \mathrm{v}) = \frac{2(2r)^{n-2}\left(1+\log\left(2+\frac{2r^2\left(\|\mathrm{u}+\mathrm{v}\|\|\mathrm{u}-\mathrm{v}\|-\left|\|\mathrm{u}\|^2-\|\mathrm{v}\|^2\right|\right)}{\delta}\right)\right)}{\sqrt{\|\mathrm{u}\|^2\|\mathrm{v}\|^2-|\mathrm{u}^\mathsf{T}\mathrm{v}|^2}\,V(n,r)} \tag{22}$$

Proof. We have

$$\begin{aligned}
&\mathrm{P}\left[|\mathbf{a}^\mathsf{T}\mathrm{C}\mathbf{a}| \le \delta\right] \\
&= \frac{1}{V(n,r)} \int_{\mathcal{B}_n(0,r)} \chi_{\left\{\mathrm{a}\in\mathbb{R}^n \,\middle|\, |\mathrm{a}^\mathsf{T}\mathrm{C}\mathrm{a}|<\delta\right\}}\, \mathrm{da} \\
&= \frac{1}{V(n,r)} \int_{\mathcal{B}_n(0,r)} \chi_{\left\{\mathrm{a}\in\mathbb{R}^n \,\middle|\, |\mathrm{a}^\mathsf{T}\mathrm{W}\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}\mathrm{W}^\mathsf{T}\mathrm{a}|<\delta\right\}}\, \mathrm{da} && (23)\\
&= \frac{1}{V(n,r)} \int_{\mathcal{B}_n(0,r)} \chi_{\left\{\mathrm{b}\in\mathbb{R}^n \,\middle|\, |\mathrm{c}^\mathsf{T}\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}\mathrm{c}|<\delta\right\}}\, \mathrm{db} && (24)\\
&\le \frac{(2r)^{n-2}}{V(n,r)} \int_{\mathcal{B}_2(0,r)} \chi_{\left\{\mathrm{c}\in\mathbb{R}^2 \,\middle|\, |\mathrm{c}^\mathsf{T}\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}\mathrm{c}|<\delta\right\}}\, \mathrm{dc} && (25)
\end{aligned}$$

where (23) follows from Lemma 4 with R and J defined in (32) and W defined in (33) and (24) follows from changing variables to $\mathrm{a} = \bar{\mathrm{W}}\mathrm{b}$ with $\bar{\mathrm{W}} = (\mathrm{W}, \mathrm{Z}) \in \mathbb{R}^{n\times n}$ where $\mathrm{Z} \in \mathbb{R}^{n\times(n-2)}$ is chosen in such a way that $\bar{\mathrm{W}}\bar{\mathrm{W}}^\mathsf{T} = \mathrm{I}$ and $\mathrm{c} = (c_1, c_2)^\mathsf{T}$ with $c_1 = b_1$ and $c_2 = b_2$.

The bound (36) on the determinant of the matrix RJR^T implies that one eigenvalue of RJR^T, say λ_1, is positive and the other eigenvalue of RJR^T, say $-\lambda_2$, is negative. We can assume without loss of generality that $\lambda_1 \ge \lambda_2$. Using the eigendecomposition $\mathrm{RJR}^\mathsf{T} = \mathrm{U}\operatorname{diag}(\lambda_1, -\lambda_2)\mathrm{U}^\mathsf{T}$, where $\mathrm{U} \in \mathbb{R}^{2\times 2}$ with $\mathrm{UU}^\mathsf{T} = \mathrm{I}$, and changing variables to $\mathrm{c} = \mathrm{Ud}$, we can further upper bound (25) by

$$\begin{aligned}
&\frac{(2r)^{n-2}}{V(n,r)} \int_{\mathcal{B}_2(0,r)} \chi_{\left\{\mathrm{c}\in\mathbb{R}^2 \,\middle|\, |\mathrm{c}^\mathsf{T}\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}\mathrm{c}|<\delta\right\}}\, \mathrm{dc} \\
&= \frac{(2r)^{n-2}}{V(n,r)} \int_{\mathcal{B}_2(0,r)} \chi_{\left\{\mathrm{d}\in\mathbb{R}^2 \,\middle|\, |\lambda_1 d_1^2-\lambda_2 d_2^2|<\delta\right\}}\, \mathrm{dd} \\
&= \frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, \frac{t_1^2}{\lambda_1}+\frac{t_2^2}{\lambda_2}\le r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, |t_1^2-t_2^2|<\delta\right\}}\, \mathrm{dt} && (26)\\
&\le \frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, t_1^2\le\lambda_1 r^2, t_2^2\le\lambda_2 r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, |t_1^2-t_2^2|<\delta\right\}}\, \mathrm{dt} && (27)
\end{aligned}$$

where in (26) we changed variables to $\mathrm{t} = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})\mathrm{d}$. The integral in (27) measures the area that is inside the rectangle $\{\mathrm{t} \mid t_1^2 \le \lambda_1 r^2, t_2^2 \le \lambda_2 r^2\}$ and the two hyperbolas $\{\mathrm{t} \mid t_1^2 - t_2^2 = \pm\delta\}$ (see Figure 1). The bound (21) can then be established by performing the following to steps:

1) deriving an upper bound on the integral in (27).
2) finding an expression of the eigenvalues of RJR^T in terms of the vectors u and v,

which will be done next. We have

$$\begin{aligned}
&\frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, t_1^2\le\lambda_1 r^2, t_2^2\le\lambda_2 r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, |t_1^2-t_2^2|<\delta\right\}}\, \mathrm{dt} \\
&\le \frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, t_1^2+t_2^2\le\delta+2\lambda_2 r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{t}\in\mathbb{R}^2 \,\middle|\, |t_1^2-t_2^2|<\delta\right\}}\, \mathrm{dt} && (28)\\
&= \frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, z_1^2+z_2^2\le\delta+2\lambda_2 r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, |z_1 z_2|<\frac{\delta}{2}\right\}}\, \mathrm{dz} && (29)\\
&\le \frac{(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2} \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, z_1^2\le\delta+2\lambda_2 r^2,\, z_2^2\le\delta+2\lambda_2 r^2\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, |z_1 z_2|<\frac{\delta}{2}\right\}}\, \mathrm{dz} \\
&= \frac{4(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}\,V(n,r)} \int_{\mathbb{R}^2_\ge} \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, z_1\le\sqrt{\delta+2\lambda_2 r^2}\right\}} \\
&\qquad\qquad \times \chi_{\left\{\mathrm{z}\in\mathbb{R}^2 \,\middle|\, z_2\le\min\left(\sqrt{\delta+2\lambda_2 r^2}, \frac{\delta}{2z_1}\right)\right\}}\, \mathrm{dz}
\end{aligned}$$

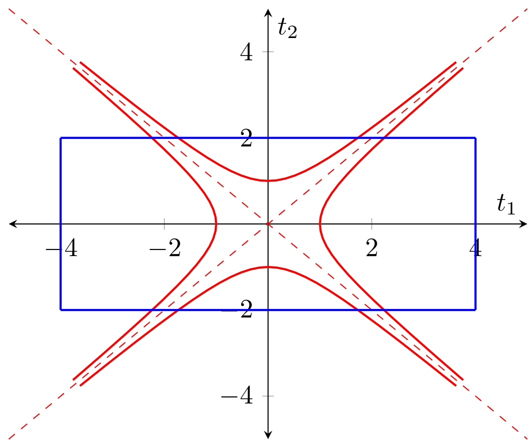

Fig. 1. Intersection of the rectangle $\{\mathrm{t} \mid t_1^2 \le \lambda_1 r^2, t_2^2 \le \lambda_2 r^2\}$ with the two hyperbolas $\{\mathrm{t} \mid t_1^2 - t_2^2 = \pm\delta\}$ for $\delta = 1$, $\lambda_1 = 16/r^2$, and $\lambda_2 = 4/r^2$.

$$\leq \frac{4(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}V(n,r)} \int_{\mathbb{R}^2_{\geq}} \chi_{\left\{z\in\mathbb{R}^2 \middle| z_1 \leq \frac{\delta}{2\sqrt{\delta+2\lambda_2 r^2}}\right\}} \times \chi_{\left\{z\in\mathbb{R}^2 \middle| z_2 \leq \sqrt{\delta+2\lambda_2 r^2}\right\}} \mathrm{d}z$$

$$+ \frac{4(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}V(n,r)} \int_{\mathbb{R}^2_{\geq}} \chi_{\left\{z\in\mathbb{R}^2 \middle| \frac{\delta}{2\sqrt{\delta+2\lambda_2 r^2}} < z_1 \leq \sqrt{\delta+2\lambda_2 r^2}\right\}} \times \chi_{\left\{z\in\mathbb{R}^2 \middle| z_2 \leq \frac{\delta}{2z_1}\right\}} \mathrm{d}z$$

$$= \frac{2\delta(2r)^{n-2}}{\sqrt{\lambda_1\lambda_2}V(n,r)}\left(1+\log\left(2+\frac{4\lambda_2 r^2}{\delta}\right)\right) \tag{30}$$

where in (28) we used that $t_2^2 \leq \lambda_2 r^2$ and $|t_1^2 - t_2^2| < \delta$ imply $t_1^2 + t_2^2 \leq \delta + 2\lambda_2 r^2$, and in (29) we applied the orthogonal transformation $z_1 = (1/\sqrt{2})(t_1+t_2)$, $z_2 = (1/\sqrt{2})(t_1-t_2)$. Combining (25) with (30) and using the expressions (36) and (38) gives (22). □

VI. Properties of certain rank two matrices

Lemma 4. *Let* $\mathrm{u}, \mathrm{v} \in \mathbb{R}^n$ *be linearly independent and* $\mathrm{C} = \mathrm{u}\mathrm{u}^\mathsf{T} - \mathrm{v}\mathrm{v}^\mathsf{T}$. *Then,*

$$\mathrm{C} = \mathrm{W}\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}\mathrm{W}^\mathsf{T} \tag{31}$$

with

$$\mathrm{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{R} = \begin{pmatrix} \|\mathrm{u}\| & \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|} \\ 0 & \|\mathrm{v} - \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2}\mathrm{u}\| \end{pmatrix} \tag{32}$$

and

$$\mathrm{W} = \left(\frac{\mathrm{a}}{\|\mathrm{a}\|}, \frac{\mathrm{b}}{\|\mathrm{b}\|}\right) \tag{33}$$

where the orthonormal vectors $\mathrm{a}/\|\mathrm{a}\|$ *and* $\mathrm{b}/\|\mathrm{b}\|$ *are defined by*

$$\mathrm{a} = \mathrm{u} \tag{34}$$

$$\mathrm{b} = \mathrm{v} - \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2}\mathrm{a}. \tag{35}$$

Moreover,

$$\det(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) = |\mathrm{u}^\mathsf{T}\mathrm{v}|^2 - \|\mathrm{u}\|^2\|\mathrm{v}\|^2 < 0 \tag{36}$$

$$\mathrm{tr}(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) = \|\mathrm{u}\|^2 - \|\mathrm{v}\|^2 \tag{37}$$

$$\sigma_2(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) = \frac{1}{2}\|\mathrm{u}+\mathrm{v}\|\|\mathrm{u}-\mathrm{v}\| - \frac{1}{2}\left|\|\mathrm{u}\|^2 - \|\mathrm{v}\|^2\right| \tag{38}$$

where $\sigma_1(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) \geq \sigma_2(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})$ *are the singular values of* $\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}$.

Proof. We can rewrite $\mathrm{C} = \mathrm{A}\mathrm{J}\mathrm{A}^\mathsf{T}$ with $\mathrm{A} = (\mathrm{u}, \mathrm{v})$. Hence, to prove (31), it is sufficient to show that $\mathrm{A} = \mathrm{W}\mathrm{R}$.

Using the definitions of the vectors a and b in (34) and (35), we can rewrite

$$\begin{aligned} \mathrm{A} &= \left(\mathrm{a}, \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2}\mathrm{a} + \mathrm{b}\right) \\ &= (\mathrm{a}, \mathrm{b}) \begin{pmatrix} 1 & \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2} \\ 0 & 1 \end{pmatrix} \\ &= \left(\frac{\mathrm{a}}{\|\mathrm{a}\|}, \frac{\mathrm{b}}{\|\mathrm{b}\|}\right) \begin{pmatrix} \|\mathrm{u}\| & \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|} \\ 0 & \|\mathrm{v} - \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2}\mathrm{u}\| \end{pmatrix} \\ &= \mathrm{W}\mathrm{R} \end{aligned}$$

which proves (31).

The explicit form of the determinant in (36) follows from the fact that

$$\begin{aligned} \det(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) &= \det(\mathrm{R})\det(\mathrm{J})\det(\mathrm{R}^\mathsf{T}) \\ &= -|\det(\mathrm{R})|^2 \\ &= -\|\mathrm{u}\|^2\left\|\mathrm{v} - \frac{\mathrm{u}^\mathsf{T}\mathrm{v}}{\|\mathrm{u}\|^2}\mathrm{u}\right\|^2 \\ &= |\mathrm{u}^\mathsf{T}\mathrm{v}|^2 - \mathrm{u}^\mathsf{T}\mathrm{u}\mathrm{v}^\mathsf{T}\mathrm{v} \\ &< 0 \end{aligned} \tag{39}$$

where (39) follows from the Cauchy-Schwarz inequality [18, Sec. 0.6.3] and u and v being linearly independent. The expression for the trace (37) follows from $\mathrm{tr}(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) = \mathrm{tr}(\mathrm{C})$. Finally, (38) follows from

$$\begin{aligned} \sigma_2(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) &= \frac{1}{2}\left(\sigma_1(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) + \sigma_2(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})\right) \\ &\quad - \frac{1}{2}\left(\sigma_1(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T}) - \sigma_2(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})\right) \\ &= \frac{1}{2}\sqrt{\mathrm{tr}(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})^2 - 4\det(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})} - \frac{1}{2}|\mathrm{tr}(\mathrm{R}\mathrm{J}\mathrm{R}^\mathsf{T})| \\ &= \frac{1}{2}\|\mathrm{u}+\mathrm{v}\|\|\mathrm{u}-\mathrm{v}\| - \frac{1}{2}\left|\|\mathrm{u}\|^2 - \|\mathrm{v}\|^2\right|. \end{aligned}$$

□

References

[1] R. P. Millane, "Phase retrieval in crystallography and optics," *J. Opt. Soc. Am. A*, vol. 7, no. 3, pp. 394–411, 1990.

[2] J. R. Fienup, J. C. Marron, T. J. Schulz, and J. H. Seldin, "Hubble space telescope characterized by using phase-retrieval algorithms," *Appl. Opt.*, vol. 32, no. 10, pp. 1747–1767, 1993.

[3] H. M. Quiney, "Coherent diffractive imaging using short wavelength light sources," *J. Mod. Opt.*, vol. 57, no. 13, pp. 1109–1149, 2010.

[4] Y. M. Bruck and L. Sodin, "On the ambiguity of the image reconstruction problem," *Opt. Comm.*, vol. 30, no. 3, pp. 304–308, 1979.

[5] R. Balan, P. Casazza, and D. Edidin, "On signal reconstruction without phase," *Appl. Comput. Harmon. Anal.*, vol. 20, no. 3, pp. 345–356, 2006.

[6] E. J. Candès and X. Li, "Solving quadratic equations via PhaseLift when there are about as many equations as unknowns," *Found. Comput. Mathematics*, vol. 14, no. 5, pp. 1017–1026, 2012.

[7] E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski, "Phase retrieval via matrix completion," *SIAM J. on Imag. Sciences*, vol. 6, no. 1, pp. 199–225, 2013.

[8] I. Waldspurger, A. d'Aspremont, and S. Mallat, "Phase recovery, MaxCut and complex semidefinite programming," *Mathematical Programming*, vol. 149, no. 1-2, pp. 47–81, 2015.

[9] P. Netrapalli, P. Jain, and S. Sanghavi, "Phase retrieval using alternating minimization," in *Advances in Neural Information Processing Systems 26*, C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Weinberger, Eds. Curran Associates, Inc., 2013, pp. 2796–2804.

[10] E. J. Candès, X. Li, and M. Soltanolkotab, “Phase retrieval via Wirtinger flow: Theory and algorithms,” *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1985–2007, April 2015.
[11] Y. C. Eldar and S. Mendelson, “Phase retrieval: Stability and recovery guarantees,” *Appl. Comput. Harmon. Anal.*, vol. 36, no. 3, pp. 473–494, 2014.
[12] M. Akçakaya and V. Tarokh, “New conditions for sparse phase retrieval,” 2014, arXiv:1310.1351 [cs.IT].
[13] Y. Wu and S. Verdú, “Rényi information dimension: Fundamental limits of almost lossless analog compression,” *IEEE Trans. Inf. Theory*, vol. 56, no. 8, pp. 3721–3748, Aug. 2010.
[14] D. Stotz, E. Riegler, and H. Bölcskei, “Almost lossless analog signal separation,” in *Proc. IEEE Int. Symp. Inf. Theory*, Istanbul, Turkey, Jul. 2013, pp. 106–110.
[15] K. Falconer, *Fractal Geometry*, 1st ed. New York, NY: Wiley, 1990.
[16] T. T. Cai and A. Zhang, “Sparse representation of a polytope and recovery in sparse signals and low-rank matrices,” *IEEE Trans. Inf. Theory*, vol. 60, no. 1, pp. 122–132, Jan. 2014.
[17] E. Riegler, D. Stotz, and H. Bölcskei, “Information-theoretic limits of matrix completion,” in *Proc. IEEE Int. Symp. Inf. Theory*, Hong Kong, China, Jun. 2015, arXiv:1504.04970 [cs.IT].
[18] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.