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The Cyclic Prefix of OFDM/DMT – An Analysis

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Abstract— **We address the impact of a too short cyclic prefix on multicarrier systems such as Orthogonal Frequency Division Multiplex (OFDM) and Discrete MultiTone (DMT). The main result is that the intersymbol interference (ISI) and intercarrier interference (ICI) may be spectrally concentrated and analytical expressions showing this are given. A practical implication is, *e.g.*, that the cyclic prefix in some xDSL systems can be surprisingly short, as shown in one example of ADSL transmission.**

Keywords— **Intersymbol interference (ISI), intercarrier interference (ICI), multicarrier transmission, cyclic prefix, time-domain equalization.**

I. Introduction

In this paper, we address the impact of a too short cyclic prefix in multicarrier systems [1]. The cyclic prefix removes the intersymbol interference (ISI) and intercarrier interference (ICI) [2]. The introduction of the cyclic prefix of length L, see Fig. 1, gives a constant capacity loss, since the channel does no longer carry data for short periods of time. As such, one would like to minimize the length of the cyclic prefix, preferably maintaining performance. Common wisdom is to choose the cyclic prefix to be of roughly the same length as the channel (or system) impulse response, thus eliminating ISI and ICI. It is also well established that, if the tail of the impulse response contains only very little energy, it has little impact and can be considered zero allowing a shorter cyclic prefix. We show that, furthermore, the ISI and ICI can be spectrally concentrated and sometimes have limited or even zero impact on performance. Although this is not completely unknown among designers of DMT digital subscriber loop (DSL) systems, it has rarely been given a thorough analysis. Our attempt here is to provide an intuitive and immediate understanding of the mechanisms involved.

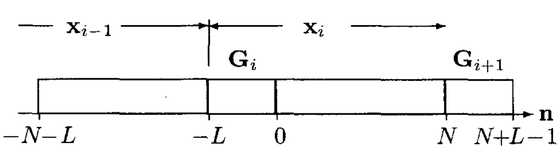

Fig. 1. Symbol structures

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We give a mathematical analysis of the ISI and ICI for a system with insufficient length of the cyclic prefix for the case when all the tones are used to transmit data in one direction, for instance in simplex communication or for a time-division duplex (TDD) system. This is done in Section II. In Section III, we present an example where the interference is strikingly concentrated in frequency.

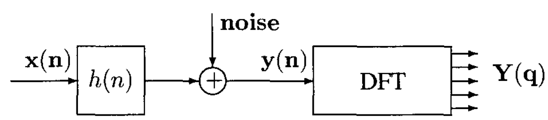

Fig. 2. The system under analysis: the receiving end of a multicarrier system

II. Signal Model and Interference Calculation

The interference that we are studying consists of two parts: the intersymbol interference (ISI) and the intercarrier interference (ICI). We start with analyzing the ISI, which can be derived in a more intuitive fashion. We will also see that the ICI is of very similar structure, and can be described with the same mathematics under our present assumptions.

Consider the transmission of symbols $x_i(n)$ through a channel with the impulse response $h(n)$ of length L_h. We extend our notations with an index i on the input sequence $x_i(n)$ as we need to distinguish between the input corresponding to the present symbol $y_i(n)$, which gives rise to the ICI, and the previous data $x_{i-1}(n)$ which causes the ISI[1]. The signal $x(n)$ is assumed to be zero mean with a variance of σ_x^2 and $x(L+1), x(L+2), \ldots, x(L_h-1)$ are assumed to be pairwise uncorrelated. The received signal, which is to be processed by the FFT,

$$y_i(n) = h(n) * x(n) = \sum_{\nu=0}^{L_h-1} h(\nu)x(n-\nu), \qquad (1)$$

[1] We do not consider the case where the evaluation frame at the receiver is positioned such that it has post- and precursors from both neighboring frames. In a real system, this could be the case. However, this would make the math lengthy and difficult to follow. Thus, in order to outline the fundamental properties, we decided to restrict this presentation to only postcursors from a preceding frame.

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where $x(n)$ denotes the concatenation of all $x_i(n)$ up to the present symbol. A part of this signal will then be ISI from the previous symbol. Figure 3 illustrates the tails of the impulse response that are not covered by the cyclic prefix. Note that in contrast to Fig. 1 not only the guard interval but also its counterpart at the end of the frame is highlighted (hatched) and labeled with **G**. The ISI that affects a symbol i from a symbol $i-1$ is

$$y_{\mathrm{ISI}}(n) = \sum_{\nu=L+1+n}^{L_h-1} h(\nu)x(n-\nu), \ \text{with } 0 \le n \le L_h - L - 2\,. \tag{2}$$

In (2), we actually used the concatenated symbols in order to simplify the description.

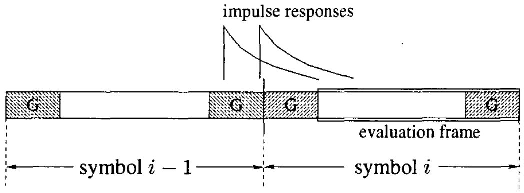

Fig. 3. ISI from symbol $i-1$ to symbol i

The signal after the DFT, $Y_i(q)$, then becomes

$$\begin{aligned} Y_i(q) &= \sum_{n=0}^{N-1} y_i(n)e^{-j\frac{2\pi}{N}nq} \\ &= \sum_{n=0}^{N-1}\sum_{\nu=0}^{L_h-1} h(\nu)x(n-\nu)e^{-j\frac{2\pi}{N}nq}. \end{aligned} \tag{3}$$

With a cyclic prefix of length L, the residual ISI is determined as

$$Y_{\mathrm{ISI}}(q) = \sum_{n=0}^{N-1}\sum_{\nu=L+1+n}^{L_h-1} h(\nu)x(n-\nu)e^{-j\frac{2\pi}{N}nq} \tag{4}$$

$$= \sum_{n=0}^{N-1}\sum_{m=L+1}^{L_h-1-n} h(m+n)x(-m)e^{-j\frac{2\pi}{N}nq}\,. \tag{5}$$

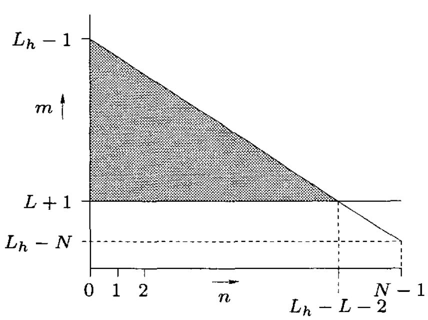

Fig. 4. Summation area in (5)

Next, we interchange the two sums. Due to the dependence of n in the inner sum, we have to investigate the summation over the pair (n, m) in some more detail. Figure 4 shows the effective pairs that are used in the two sums. There, we assume that $L_h - N \le L + 1$, i.e., $N \ge L_h - L - 1$. This is a reasonable assumption. One would certainly not design a DMT (OFDM) system with a frame length shorter than the channel impulse response. While n is increased, the upper limit of the inner sum $L_h - 1 - n$ is decreased accordingly. The lower limit $L+1$ is reached ($L_h - 1 - n = L+1$) when $n = L_h - L - 2$. If we now take the sum over m as outer sum, we obtain the upper limit for the inner sum over n as $L_h - L - 2 - (m-(L+1)) = L_h - m - 1$ and thus,

$$Y_{\mathrm{ISI}}(q) = \sum_{m=L+1}^{L_h-1} x(-m) \sum_{n=0}^{L_h-m-1} h(m+n)e^{-j\frac{2\pi}{N}nq}\,. \tag{6}$$

Substituting $m + n = \mu$ yields

$$\begin{aligned} Y_{\mathrm{ISI}}(q) &= \sum_{m=L+1}^{L_h-1} x(-m) \sum_{\mu=m}^{L_h-1} h(\mu)e^{-j\frac{2\pi}{N}(\mu-m)q} \\ &= \sum_{m=L+1}^{L_h-1} x(-m)e^{+j\frac{2\pi}{N}mq}\,. \\ &\quad \cdot \underbrace{\sum_{\mu=m}^{L_h-1} h(\mu)e^{-j\frac{2\pi}{N}\mu q}}_{=:H_m(q)}\,. \end{aligned} \tag{7}$$

Note that the expression for $H_m(q)$ is actually the DFT of the tail of the impulse-response. The power spectral density $N_{\mathrm{ISI}}(q)$ after the FFT due to the intersymbol interference follows to be

$$N_{\mathrm{ISI}}(q) = E\{Y_{\mathrm{ISI}}(q)Y^*_{\mathrm{ISI}}(q)\} \tag{8}$$

$$= \sum_{m_1=L+1}^{L_h-1}\sum_{m_2=L+1}^{L_h-1} H_{m_1}(q)H^*_{m_2}(q)$$

$$E\{x(-m_1)x^*(-m_2)\}\, e^{j\frac{2\pi}{N}(m_1-m_2)q} \tag{9}$$

$$= \sigma_x^2 \sum_{m=L+1}^{L_h-1} |H_m(q)|^2 \tag{10}$$

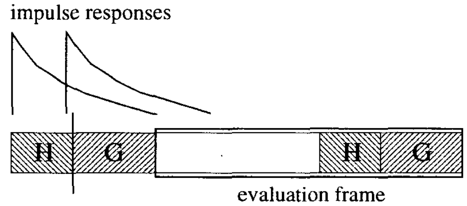

Fig. 5. Hypothetic extension "H" of the guard interval "G" for computing the ICI

Remains now to calculate the ICI term. ICI would not be an issue if the guard interval would be big enough to fake the channel convolution as a cyclic convolution within the frame that is evaluated by the receiver. Thus we just assume an extension of the guard interval that would null the ICI and compute its impact on the evaluation frame. Its negative would then be the ICI signal. The hypothetic frame extension is shown in Fig. 5. The negated ICI signal is then similar to (2) and can be written as

$$y_{\text{ICI}}(n) = -\sum_{\nu=L+1+n}^{L_h-1} h(\nu) x_i\left((n-\nu) \bmod N\right), \qquad (11)$$

$$0 \leq n \leq L_h - L - 2\,.$$

Note that this time, (11) refers to only one input signal block x_i from which the hypothetical guard interval is taken. Correspondingly, we count modulo N to stay within this frame. In DFT domain, we obtain

$$Y_{\text{ICI}}(q) = -\sum_{n=0}^{L_h-L-1} \sum_{\nu=L+1+n}^{L_h-1} h(\nu) x_i\left((n-\nu) \bmod N\right) e^{-j\frac{2\pi}{N}nq} \qquad (12)$$

which corresponds to (4). If we further follow the steps in the derivation of the ISI result down to (10), we see that the only differences are the minus sign and that we count modulo N, staying within the same frame. The minus sign disappears with the squaring when computing the power spectral density. If we consider x_i instead of x, does not change the result, either, so that we obtain

$$N_{\text{ICI}}(q) = N_{\text{ISI}}(q) \qquad (13)$$

and

$$N_{\text{ICI+ISI}} = 2 \cdot \sigma_x^2 \sum_{m=L+1}^{L_h-1} |H_m(q)|^2\,. \qquad (14)$$

ISI and ICI have the same power spectral density. This is an important first result that has relevance for practical time-domain equalizer algorithms (see, e.g., [3]).

III. Example

As a practical example we choose an ADSL transmission with an FFT length of $N = 512$ and a guard interval of $L = 32$ samples. We select the impulse response of a 4 km long loop of German 0.4 mm cables, which is shown in Fig. 6, cutting non-causal precursors that are due to the underlying cable model. We see that it certainly has a portion exceeding the guard interval. If we compute the ICI and ISI components according to (10) and (13), we see in Fig. 7 that the noise is predominantly disturbing low-frequency components. It has long been realized that the signal-to-noise ratio decreases near DC, which was considered partly to be due to the leakage effect of the DFT which folds high-frequency noise components into the low-frequency range. The results in here now show that noise due to ISI and ICI is concentrated around DC as well.

IV. Conclusions

We derived closed formulas for the intersymbol and interchannel interference power spectral density and found that they are actually the same. From a typical example we concluded that this noise will be concentrated around DC.

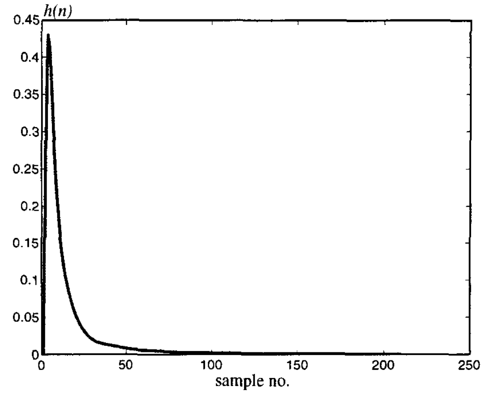

Fig. 6. Impulse response of a 4 km, 0.4 mm loop

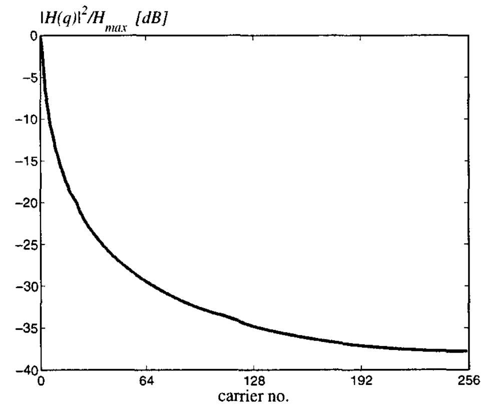

Fig. 7. ISI and ICI power spectral density according to (10) and (13) normalized to its maximum ($|H(q)|^2 = \sum_{m=L+1}^{L_h-1} |H_m(q)|^2$)

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