Lossless Analog Compression

Abstract

We establish the fundamental limits of lossless analog compression by considering the recovery of random vectors $\mathbf{x} \in \mathbb{R}^m$ from the noiseless linear measurements $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ with measurement matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$. Specifically, we show that, for Lebesgue-almost all $\boldsymbol{A}$, $\mathbf{x}$ can be recovered with zero-error probability provided that $n > K(\mathbf{x})$, where the description complexity $K(\mathbf{x})$ is given by the infimum of the lower modified Minkowski dimensions over all support sets $\mathcal{U}$ of $\mathbf{x}$ (sets $\mathcal{U} \subseteq \mathbb{R}^m$ with $\mathrm{P}[\mathbf{x} \in \mathcal{U}] = 1$). Lower modified Minkowski dimension gereralizes the concept of s-sparsity as it allows to consider support sets that are not necessarily countable unions of s-dimensional linear subspaces. We particularize our achievability result to s-rectifiable random vectors $\mathbf{x}$ introduced in Koliander *et al.* (2016), which are random vectors of absolutely continuous distributions with respect to the s-dimensional Hausdorff measure admitting support sets that are countable unions of s-dimensional C^1-submanifolds of $\mathbb{R}^m$, showing that they can be recovered with zero error probability provided that $n > s$. Again, this statement holds for Lebesgue-a.a. measurement matrices. We also provide several examples of s-rectifiable random vectors. In particular, we give an example of an s-rectifiable random vector that can be recovered with zero-error probability from $n < s$ linear measurements. Finally, we introduce the class of s-analytic random vectors, for which we state a strong converse in the sense of $n \geq s$ being necessary for recovery with probability of error smaller than one.