Convergence results for linear multistep methods for quasi-singular perturbed problem systems

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Abstract

Stiff behavior occurs in a variety of ODE systems relevant in applications. The notion of stiffness is a phenomenological one, and a stability and error analysis of numerical methods has been based either on simple models or particular problem structures. In particular, stiff initial value problems in standard singular perturbation form are well understood. However, problems of this type exhibit a very simple phase space geometry, namely the stiff eigendirections also behave *stiff* in another sense, i.e., they are almost parallel. This motivates us to consider a more general nonlinear class of stiff ODE systems depending on a small parameter. In particular, we investigate the convergence properties of the implicit Euler scheme applied to problems of this type, and linear multistep schemes will further be investigated.

1 The class of quasi-singular perturbed problems

The idea of extending stability theory by considering a more general class of problems (not only of singular perturbed form) is not new, but has already been done by Auzinger, Schranz-Kirlinger and Frank (see [1]). In this paper, a new ansatz, basing on a generalised form of singular pertubation, is introduced.

Definition 1 *We call an ODE sytem a singular perturbed (SPP) problem iff it takes the form*

$$\dot{z}_1 = F^{\flat}(z_1, z_2), \qquad (1)$$

$$\dot{z}_2 = \frac{1}{\epsilon} F^{\sharp}(z_1, z_2), \qquad (2)$$

where the real part of the spectrum of the Jacobian matrix $D_{z_2} F^{\sharp}$ is smaller than some $-b_0$ for $b_0 > 0$.

There is a well developped theory about the dynamics of SPP, basing on the works of Fenichel [2].

In [3] Nipp has shown that under some reasonable assumptions, SPP have a strongly attractive smooth invariant manifold with moderate flow.

Definition 2 *A system $\dot{y} = G(y)$ is called quasi-singular perturbed (QSPP) iff there exists a transformation $y = \Phi(z)$ with $\|D\Phi\| = O(1)$ and $\|D(\Phi^{-1})\| = O(1)$ which transforms the system to singular perturbed form.*

Lemma 1 *QSPP have a strongly attractive smooth invariant manifold with moderate flow.*

Proof: This is simply due to the moderate Lipschitz constants of the transformation. Let $\hat{\Psi}_t$ be the flow of the quasi-singular perturbed system, let Ψ_t be the flow of the SPP, and Φ be the transformation onto SPP form. Then we have $\hat{\Psi}_t = \Phi \circ \Psi_t \circ \Phi^{-1}$, and the claimed properties are deduced easily.

Theorem 1 *If an ODE system $\dot{y} = G_\epsilon(y)$ has an invariant manifold $\mathcal{M}$ which is strongly attractive and has moderate flow, then there locally exists a transformation with moderate Lipschitz constants onto singular perturbed form.*

Proof(Sketch): We construct a flow-invariant foliation, which induces a transformation. We show that the transformation is moderate and yields a SPP. We consider a system $y' = G^\flat(y) + \frac{1}{\epsilon}G^\sharp(y)$ of dimension n with strongly attractive invariant k-manifold with moderate flow. Let $\hat{\Psi}_t$ be the flow of the system. We consider a smooth transversal $(n-1)$-manifold $\mathcal{N}$ with the following properties:

(i) $\mathcal{N}$ is being parametrized by a function $s(\theta, \xi) : \mathbb{R}^l \times \mathbb{R}^k \to \mathcal{N}$ (with $l + k = n - 1$) such that $s(\theta = 0, -) = \mathcal{N} \cap \mathcal{M}$, and $s(0, -)$ is a parametrisation of $\mathcal{N} \cap \mathcal{M}$. The Jacobian $Ds(y)$ is moderate with moderate inverse for every $y \in \mathcal{N}$.

(ii) $G^\sharp(y) \in \mathrm{span}(\frac{ds}{d\theta_i})_{i=1,\dots,l}$ for every $y \in \mathcal{N}$.

The existence of $\mathcal{N}$ is easily shown provided the smoothness of the manifold. Under the above conditions, it can be shown that the local foliation parametrized by $s^*(t, \theta, \xi) = \Psi_t(s(\theta, \xi))$ gives a transformation on SPP form with the desired properties. Due to property (ii) and the moderate flow on the manifold the transformation separates the $O(1/\epsilon)$-dependency, while the strong attractivity of $\mathcal{M}$ implies the eigenvalue condition from Definition 1. Condition (i) gives the moderate Lipschitz-continuity of the transformation and its inverse.

2 Multistep methods applied to stiff problems

2.1 Auxiliary results

We need to derive some auxiliary results for the subsequent chapter.

Theorem 2 *We apply a BDF method of order k to a quasi-singular perturbed problem $y' = G(y)$ with transformation $y = \Phi(z)$ on SP form:*

$$\sum_{i=0}^{k} \alpha_i y_i = h \sum_{i=0}^{k} \beta_i G(y_i). \tag{3}$$

We define $z_i = \Phi^{-1}(y_i)$ and $Z_i = (z_{i-k}, \dots z_{i-1})$. Let the starting values $(z_i)_{i=1,\dots k} = ((z_{i,1}, z_{i,2}))_{i=1,\dots k}$ be such that for the point z_* given by

$$\Phi(z_*) := \alpha_k^{-1} \sum_{i=1}^{k-1} \alpha_i \Phi(z_i)$$

there are functions θ_i such that $\theta_i(Z_i, h)$ such that $z_{i,1} = z_{*,1} + h\theta_i(Z_i, h)$. Let be $\|z_{i,2} - s(z_{i,1})\| \leq d$. and $\mathcal{M}$ be the invariant smooth manifold *of the dynamical system, with parameterization $\mathcal{M} = \{(x, s(x)\}$. Define $U_d := \{|y - s(x)| \leq d.\}$. Then, the BDF step lies within an $O(d + h)$-neighbourhood of the manifold.*

Proof: The proof is merely technical. As the BDF step is the root of the system of equations $P(z_k) = 0$, $Q(z_k) = 0$ with

$$P(z_k) = \pi_\flat \left(D\Phi(z_k)^{-1} (\sum_{i=0}^{k} \alpha_i \Phi(z_i)) \right) - hF^\flat(z_k), \quad (4)$$

$$Q(z_k) = \frac{\epsilon}{h} \pi_\sharp \left(D\Phi(z_k)^{-1} (\sum_{i=0}^{k} \alpha_i \Phi(z_i)) \right) - F^\sharp(z_k), \quad (5)$$

where $\pi_\flat$ and $\pi_\sharp$ are the canonical linear projections onto smooth and stiff component, we can apply the Newton-Kantorovich theorem to derive estimates for the position of z_k. Details can be found in [7].

2.2 Main results

The main results of the theory of multistep methods (LMM) applied to stiff problems are due to Lubich [4] as well as Nipp and Stoffer [5]. We give the main theorem for reasons of readability.

Theorem 3 *Let $(\alpha_i, \beta_i)_{(0 \leq i \leq k)}$ be an $A(\alpha)$-stable LMM of order p. When applied to the SPP $\dot{z} = F(z)$ Then, the error for $mh < \bar{t} - t_0$ is bounded by*

$$\begin{aligned} \|y_m - \hat{y}_m\| \quad &\leq \quad C \left(\max_{0 \leq j < k} \|y_j^\flat - \hat{y}_j^\flat\| + h^p \int_{t_0}^{t_m} \| \frac{d^{p+1}}{ds^{p+1}} y^\flat(s) \| ds \right. \\ &+ \quad \left. (h + \rho^m) \max_{0 \leq j < k} \|y_j^\sharp - \hat{y}_j^\sharp\| + \epsilon h^p \max_{t_0 < t < t_m} \left\| \frac{d^{p+1}}{ds^{p+1}} y^\sharp(s) \right\| \right) \end{aligned} \quad (6)$$

This estimate holds for $h < h_0$ for sufficiently small h_0 independly of ϵ, for $\epsilon \leq ch$, for sufficiently small c and assumed the distance of the starting values $y_j, 0 \leq j < k$ to the smooth invariant manifold $\mathcal{M}$ is suffiently small (smaller than some d).

Proof: see [4].

Since we have only been able to prove the auxiliary result for BDF methods, we have to restrict on that case in the generalisation; the concept of discrete variation of constants is powerful enough to prove the result for any LMM.

Theorem 4 *Let $(\alpha_i, \beta_i)_{(0 \leq i \leq k)}$ be an $A(\alpha)$-stable BDF method of order p. Applied to the QSPP $\dot{y} = G(y)$ which is transformed onto an SPP fulfilling the assumptions of Theorem 3, the same convergence result as in Theorem 3 holds, provided $h, d, \bar{t}$ and ϵ are small enough ($h, \bar{t}$ and d independently of ϵ.)*

Proof (Sketch): To a sequence of BDF steps y_m we define the associated sequence z_m where $y_m = \Phi(z_m)$. We can decompose $G(y) = G^\flat(y) + \frac{1}{\epsilon} G^\sharp(y)$; since $G(y) = D\Phi(z) F(\Phi^{-1}(z))$ and $G^\sharp(\mathcal{M}) = 0$ on the smooth invariant manifold $\mathcal{M}$, we have that

$$DG^\sharp(y_m) = D\Phi(z_m) F(\Phi^{-1}(z_m)) D(\Phi^{-1}(z_m)) + O(d) + O(h)$$

from Theorem 2. So, we get that the Jacobian of a QSPP is similar to the Jacobian of a SPP up to perturbations, implying that the spectra are the same up to perturbations. So, for h and d sufficiently small, spec(DG) can be decomposed into some stiff eigenvalues $O(1/\epsilon)$ which form a subset of the *stability cone* $\{\lambda \in \mathbb{C},\ \arg(\lambda - \pi) < \alpha\}$ of the applied BDF. We decompose the errors $e_m := e_m^\flat + e_m^\sharp$, $d_m := d_m^\flat + d_m^\sharp$. If the perturbations of the spectra are small enough,we can apply the results of Lubichs discrete variation of constants theory (see [6]) to get the following estimates for the error components $e_m^\flat$ and $e_m^\sharp$:

$$\|e_m^\flat\| \leq h\sum_{j=0}^{m}\Big(K_\flat\|e_j^\flat\| + K_\sharp\|e_j^\sharp\|\Big) + \sum_{j=0}^{m}\|d_j^\flat\|, \tag{7}$$

$$\|e_m^\sharp\| \leq \sum_{j=0}^{m}\Big(L_\flat\kappa^{m-j}\|e_j^\flat\| + L_\sharp\kappa^{m-j}\|e_j^\sharp\|\Big) + \frac{\epsilon}{h}\sum_{j=0}^{m}C\kappa^{m-j}\|d_j^\sharp\|. \tag{8}$$

where $L_\sharp$ can be made arbitrarily small by reducing the stepsize h, the distance d, the length of the considered interval $\bar{t}$ (here we need Theorem 2). We define the sequences u and v recursively by $u_0 := \|e_0^\flat\|$, $v_0 := \|e_0^\sharp\|$ and

$$\begin{pmatrix} u_m \\ v_m \end{pmatrix} = \begin{pmatrix} u_{m-1} \\ \kappa v_{m-1} \end{pmatrix} + \begin{pmatrix} hK_\flat & hK_\sharp \\ L_\flat & L_\sharp \end{pmatrix}\begin{pmatrix} u_m \\ v_m \end{pmatrix} + \begin{pmatrix} \|d_m^\flat\| \\ C\frac{\epsilon}{h}\|d_m^\sharp\| \end{pmatrix} \tag{9}$$

such that $u_m \geq \|e_m^\flat\|$ and $v_m \geq \|e_m^\sharp\|$ holds for every m. Provided that $L_\sharp$ is small enough, we can we deduce properties of the sequence $(u,v)^T$. Together with estimates for the local errors $d^\flat$, $d^\sharp$ this yields the claimed convergence result.

References

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