A Nicholson-type integral for the cross-product of the Bessel functions

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Abstract
We prove a new Nicholson-type integral representation for the cross-product of the Bessel functions $J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z)$, and related integral representations for $|H_\nu^{(1)}(z)|^2$, where $H_\nu^{(1)}$ is the Hankel function of the first kind.

Key words: Nicholson’s formula, cross-product, Bessel functions, Hankel functions

1. Introduction
The main result of this paper is the following representation of the cross-product of the Bessel functions $J_\nu$ and $Y_\nu$

\[
J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = \frac{2}{\pi} \int_C J_0(\sqrt{z^2 + w^2 - 2zw \cosh \zeta}) \cosh \nu \zeta \, d\zeta, \tag{2}
\]

where $C$ is any contour starting at 0 and ending at $\log z - \log w$. The formula holds for every complex number $\nu$, and all complex numbers $z$ and $w$ such that $|\text{Arg } z| < \pi$ and $|\text{Arg } w| < \pi$. We demonstrate it in Theorem 2.1 by verifying that the integral satisfies Bessel’s differential equation and appropriate limiting conditions at the origin.
The cross-products of the Bessel functions have long been studied theoretically, see e.g. [13, 10.6, 10.21], [8], [7]. They have also found several applications to wave propagation in cylindrical geometries, e.g. to acoustic modeling of ducts [5], and to modeling of optical waveguides [6], [14]. Other applications include heat transport [2] and free elastic oscillations of a spherical body [10]. Other integral representations of various products of the Bessel functions are well-known, an extensive list is given in [9, p. 93-98], see also [11].

Our second result is an integral representation of the square of the modulus of the Hankel function $H^{(1)}_{\nu}(z)$ of the first kind

$$|H^{(1)}_{\nu}(x+iy)|^2 = \frac{8}{\pi^2} \int_0^\infty K_0 \left( \sqrt{2x^2 \sin^2 t + y^2 \cosh^2 t} \right) \cosh 2\nu t \, dt. \quad (3)$$

This formula is valid if all $x, y, \nu$ are real, and $y > 0$. We present two simple derivations: one in Theorem 3.1 is based on a known representation of the product of the modified Bessel functions $K_{\nu}(\alpha)K_{\nu}(\beta)$, and another one in Appendix uses the Mehler-Sonine integrals.

Formula (3) has some interesting consequences. For example, it follows from (3) that for a fixed $z$ in the upper half-plane, the magnitude $|H^{(1)}_{\nu}(z)|$ increases with the order $\nu$ when $\nu \geq 0$. This fact has been observed experimentally [4], [1], but no proofs are given there. The property is needed in the numerical evaluation of the Bessel and the Hankel functions using a three-term recurrence. For the sake of numerical stability, the recurrence must proceed in the direction of increasing magnitudes. It is known that the magnitudes of the modified Bessel function $|K_{\nu}(\zeta)|$ grow with $\nu \geq 0$ [13, 10.37] when $\zeta$ is in the right half-plane, and monotonicity of $|H^{(1)}_{\nu}(z)|$ can also be deduced from this fact.

In 1910, J. W. Nicholson published [12] the following equality

$$J_{\nu}^2(z) + Y_{\nu}^2(z) = \frac{8}{\pi^2} \int_0^\infty K_0(2z \sinh t) \cosh 2\nu t \, dt, \quad (4)$$

valid for Re($z$) > 0 and an arbitrary complex order $\nu$. Nicholson’s formula can be recovered from (3) by fixing $x > 0$, letting $y \to 0^+$, and then taking analytic continuation first in $z$, and then in $\nu$. This derivation of (4) appears somewhat simpler than the one presented in [15], while another short proof is given in [16].

Finally, in Theorem 3.2 we combine (1) and (3) to obtain the following formula

$$|H^{(1)}_{\nu}(x-iy)|^2 = \frac{8}{\pi^2} \int_0^\infty K_0 \left( \sqrt{2x^2 \sin^2 t + y^2 \cosh^2 t} \right) \cosh 2\nu t \, dt \quad (5)$$

$$+ \frac{8}{\pi} \int_{\arccot \frac{y}{x}}^{\frac{\pi}{2}} I_0 \left( 2\sqrt{-x^2 \sin^2 \phi + y^2 \cos^2 \phi} \right) \cos 2\nu \phi \, d\phi,$$

which is valid if all $x, y, \nu$ are real, and $y > 0$. 2
2. Integral representation of the cross-product

The following theorem, which gives an integral representation of the cross-product of the Bessel functions $J_\nu$ and $Y_\nu$, is the main result of this paper.

**Theorem 2.1.** For all complex numbers $z$ and $w$ such that $|\text{Arg } z| < \pi$ and $|\text{Arg } w| < \pi$, and for every complex number $\nu$,

\[
J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = 2\pi \int_C J_0\left(\sqrt{z^2 + w^2 - 2zw \cosh \zeta}\right) \cosh \nu \zeta \, d\zeta,
\]

where $C$ is any contour starting at 0 and ending at $\log z - \log w$.

In this paper, $J_\nu$, $Y_\nu$ and log always denote the principal branches of these functions. Arg $z$ denotes the principal argument of a complex number $z \neq 0$. The function $J_0(\sqrt{\cdot})$ is entire, so the integral does not depend on the choice of a contour joining 0 to $\log z - \log w$.

**Proof.** Let $\alpha(z, w)$ denote the expression in (7), i.e.

\[
\alpha(z, w) = \frac{2}{\pi} \int_C J_0\left(\sqrt{z^2 + w^2 - 2zw \cosh \zeta}\right) \cosh \nu \zeta \, d\zeta.
\]

Since the integrand in (8) is symmetric with respect to $z$ and $w$, and the contour joins 0 to $\log z - \log w$, we infer that

\[
\alpha(w, z) = -\alpha(z, w).
\]

It is shown in Lemma 4.1 (see Appendix) that for a fixed $w \neq 0$, $\alpha(z, w)$ satisfies Bessel’s equation with respect to $z$ in the region $|\text{Arg } z| < \pi$. Since $J_\nu$ and $Y_\nu$ are linearly independent solutions of Bessel’s equation \cite[8.474]{3}, we have

\[
\alpha(z, w) = c_1(w)J_\nu(z) + c_2(w)Y_\nu(z),
\]

for some functions $c_1$ and $c_2$. Combining this with (9), we obtain

\[
c_1(z)J_\nu(w) + c_2(z)Y_\nu(w) = -c_1(w)J_\nu(z) - c_2(w)Y_\nu(z).
\]

Let $w_1$ and $w_2$ be any complex numbers such that $J_\nu(w_1)Y_\nu(w_2) - Y_\nu(w_1)J_\nu(w_2) \neq 0$. Setting $w = w_1$ and $w = w_2$ in (11), we obtain a system of two linear equations for $c_1(z)$ and $c_2(z)$. Therefore both $c_1(z)$ and $c_2(z)$ are linear combinations of $J_\nu(z)$ and $Y_\nu(z)$ with coefficients that do not depend on $z$ or $w$. Consequently, (10) implies that

\[
\alpha(z, w) = a_{11}J_\nu(w)J_\nu(z) + a_{21}Y_\nu(w)J_\nu(z) + a_{12}J_\nu(w)Y_\nu(z) + a_{22}Y_\nu(w)Y_\nu(z),
\]

\[12\]
where the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ do not depend on $z$ or $w$, but may depend on $\nu$. Due to linear independence of $J_\nu$ and $Y_\nu$, we infer from (9) that $a_{11} = a_{22} = 0$, and $a_{21} = -a_{12}$.

We determine the remaining coefficient $a_{12}$ by comparing the limits of (6) and (7) as $z$ and $w$ approach 0 through the positive real values with $\frac{z}{w} = r$, where $r > 0$. From (8), we obtain for $\nu \neq 0$

$$\lim_{w \to 0^+} \alpha(rw, w) = \frac{2}{\pi} J_0(0) \int_0^{\log r} \cosh \nu \zeta d\zeta = \frac{1}{\pi \nu} \left( r^\nu - r^{-\nu} \right).$$  

(13)

For $\nu = 0$, the above limit equals $\frac{2}{\pi} \log r$. The same values are obtained in Lemma 4.2 (see Appendix) for the corresponding limit of the cross-product. Therefore $a_{12} = 1$, and the proof is complete.

3. Integral representations of $|H^{(1)}_\nu(z)|^2$

In this section, we assume that the order $\nu$ is real. We consider the cases when $\text{Im}(z) > 0$ and $\text{Im}(z) < 0$ separately.

**Theorem 3.1.** If $x$ is a real number and $y > 0$, then

$$|H^{(1)}_\nu(x + iy)|^2 = \frac{8}{\pi^2} \int_0^\infty K_0 \left( \sqrt{x^2 \sinh^2 t + y^2 \cosh^2 t} \right) \cosh 2\nu t \, dt,$$  

(14)

for every real order $\nu$.

$K_\nu$ denotes the modified Bessel function of the second kind and order $\nu$, and $H^{(1)}_\nu$ denotes the Hankel function of the first kind and order $\nu$.

**Proof.** The following formula, valid when $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$, is a special case of [3, 6.648] (with $\rho$ replaced by $\nu$, and $x$ replaced by $s$)

$$K_\nu(\alpha) K_{-\nu}(\beta) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\nu s} K_0 \left( \sqrt{\alpha^2 + \beta^2 + 2\alpha \beta \cosh s} \right) \, ds.$$  

(15)

Since $K_\nu = K_{-\nu}$ [13, 10.27.3], we have

$$K_\nu(\alpha) K_{\nu}(\beta) = \frac{1}{2} \int_{-\infty}^{\infty} K_0 \left( \sqrt{\alpha^2 + \beta^2 + 2\alpha \beta \cosh s} \right) \cosh \nu s \, ds$$  

(16)

$$= \int_0^{\infty} K_0 \left( \sqrt{\alpha^2 + \beta^2 + 2\alpha \beta \cosh s} \right) \cosh \nu s \, ds$$  

(17)

$$= 2 \int_0^{\infty} K_0 \left( \sqrt{\alpha^2 + \beta^2 + 2\alpha \beta \cosh 2t} \right) \cosh 2\nu t \, dt.$$  

(18)
Since $\nu$ is real, we have [13, 10.11.9]

\[ H^{(1)}_{\nu}(z) = H^{(2)}_{\nu}(\overline{z}). \]  

(19)

From connection formulas between $K_{\nu}$ and $H^{(1)}_{\nu}, \ H^{(2)}_{\nu}$ [13, 10.27.8], we infer that

\[ H^{(1)}_{\nu}(z)H^{(2)}_{\nu}(w) = \frac{4}{\pi^2} K_{\nu}(-iz)K_{\nu}(iw), \]  

(20)

when $\text{Im}(z) > 0$ and $\text{Im}(w) < 0$. Combining (19) and (20) (with $w = \overline{z}$), we obtain

\[ |H^{(1)}_{\nu}(z)|^2 = H^{(1)}_{\nu}(z)H^{(2)}_{\nu}(\overline{z}) = \frac{4}{\pi^2} K_{\nu}(-iz)K_{\nu}(iz). \]  

(21)

The claim now follows by setting $z = x + iy, \ \alpha = -iz, \ \beta = iz$ in (18).

Another, more direct proof of this theorem is given in Appendix.

**Theorem 3.2.** If $x$ is a real number and $y > 0$, then

\[ |H^{(1)}_{\nu}(x - iy)|^2 = \frac{8}{\pi^2} \int_0^\infty K_0 \left(2\sqrt{x^2 \sin^2 t + y^2 \cosh^2 t}\right) \cosh 2\nu t \, dt \]  

(22)

\[ + \frac{8}{\pi} \int_{\arccot \frac{x}{y}}^\frac{\pi}{2} I_0 \left(2\sqrt{-x^2 \sin^2 \phi + y^2 \cos^2 \phi}\right) \cos 2\nu \phi \, d\phi \]  

(23)

for every real order $\nu$.

$I_0$ denotes the modified Bessel function of the first kind and order 0, and $\text{arccot}$ denotes the inverse function of the cotangent $\cot \phi, 0 < \phi < \pi$.

**Proof.** Replacing $z$ with $\overline{z}$ in (19), we obtain

\[ H^{(1)}_{\nu}(\overline{z}) = H^{(2)}_{\nu}(z). \]  

(24)

Combining this with (21) and connection formulas between the Hankel and the Bessel functions [13, 10.4.3], we have

\[ |H^{(1)}_{\nu}(\overline{z})|^2 = H^{(1)}_{\nu}(\overline{z})H^{(2)}_{\nu}(z) - H^{(1)}_{\nu}(z)H^{(2)}_{\nu}(\overline{z}) + |H^{(1)}_{\nu}(z)|^2 \]  

(25)

\[ = -2i (J_{\nu}(\overline{z})Y_{\nu}(z) - Y_{\nu}(\overline{z})J_{\nu}(z)) + |H^{(1)}_{\nu}(z)|^2. \]  

(26)

We set $z = x + iy, \ w = x - iy$, and express the above cross-product by means of Theorem 2.1. Choosing the contour $C$ to be the line interval joining 0 to
log $z - \log w = 2t \arccot \frac{y}{y}$, we obtain

$$J_{\nu}(\pi)Y_{\nu}(z) - Y_{\nu}(\pi)J_{\nu}(z) =$$

$$=\frac{2}{\pi} \int_{0}^{2 \arccot \frac{y}{y}} J_{0} \left(\sqrt{z^2 + w^2 - 2zw \cosh(\nu t)}\right) \cosh(\nu t) \, dt$$

$$=\frac{4t}{\pi} \int_{0}^{\arccot \frac{y}{y}} J_{0} \left(\sqrt{2x^2 - 2y^2 - 2(x^2 + y^2) \cos(2\phi)}\right) \cos(2\nu \phi) \, d\phi$$

$$=\frac{4t}{\pi} \int_{0}^{\arccot \frac{y}{y}} J_{0} \left(2 \sqrt{x^2 \sin^2 \phi - y^2 \cos^2 \phi}\right) \cos 2\nu \phi \, d\phi.$$  \hspace{1cm}(30)

The claim now follows by substituting the above and (14) into (26), and using a connection formula between $J_0$ and $I_0$ $[13, 10.27.6]$.  \hfill \Box

Since $|H^{(2)}_{\nu}(z)| = |H^{(1)}_{\nu}(\pi)|$, see (24), both theorems have their counterparts for $|H^{(2)}_{\nu}(z)|^2$.

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4. Appendix

The following two lemmas are used in the proof of Theorem 2.1.

**Lemma 4.1.** For a fixed $w \neq 0$, the function

$$y(z) = \int_{C} J_{0} \left(\sqrt{z^2 + w^2 - 2zw \cosh \zeta}\right) \cosh \nu \zeta \, d\zeta,$$  \hspace{1cm}(31)

where $C$ is any contour joining 0 to log $z - \log w$, satisfies Bessel’s differential equation

$$(zy')' + \left(z - \frac{\nu^2}{z}\right) y = 0$$  \hspace{1cm}(32)

in the region $|\text{Arg} z| < \pi$.

**Proof.** We introduce the function $u(\xi) = J_{0}(\sqrt{\xi})$, which is entire, and satisfies the following differential equation $[3, 8.491(9)]$

$$\xi u'' + u' + \frac{1}{4} u = 0.$$  \hspace{1cm}(33)
We write the contour integral (31) as follows

\[ y(z) = \int_C u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \cosh \nu \zeta \, d\zeta. \]  

(34)

Differentiating this formula under the integral sign with respect to \( z \), we obtain

\[
y'(z) = \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z - 2w \cosh \zeta) \cosh \nu \zeta \, d\zeta + [u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \cosh \nu \zeta]_{\zeta = \log z - \log w} \left( \log z - \log w \right)'
\]

(35)

\[
= \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z - 2w \cosh \zeta) \cosh \nu \zeta \, d\zeta
\]

(36)

\[ + u(0) \cosh \nu (\log z - \log w) \cdot \frac{1}{z}. \]

(37)

Consequently,

\[
zy'(z) = \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z^2 - 2zw \cosh \zeta) \cosh \nu \zeta \, d\zeta
\]

(38)

\[ + u(0) \cosh \nu (\log z - \log w). \]

(39)

Differentiating again, we obtain

\[
(zy')'(z) = \int_C \left[ u'' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z^2 - 2zw \cosh \zeta) \cosh \nu \zeta \right.
\]

(40)

\[ + u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z^2 - 2zw \cosh \zeta) \cosh \nu \zeta \, d\zeta
\]

(41)

\[ + u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (2z^2 - 2zw \cosh \zeta) \cosh \nu \zeta \] \[\zeta = \log z - \log w \left( \log z - \log w \right)'
\]

(42)

\[ + u(0) \sinh \nu (\log z - \log w) \cdot \frac{\nu}{z}.
\]

(43)

Collecting the terms, we arrive at

\[
(zy')'(z) = \int_C \left[ u'' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (4z^3 - 8z^2 w \cosh \zeta + 4zw^2 \cosh^2 \zeta) \cosh \nu \zeta \right.
\]

(44)

\[ + u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) (4z^2 - 2w \cosh \zeta) \cosh \nu \zeta \, d\zeta
\]

(45)

\[ + u'(0) (z^2 - w^2) \cosh \nu (\log z - \log w) \cdot \frac{1}{z}
\]

(46)

\[ + u(0) \sinh \nu (\log z - \log w) \cdot \frac{\nu}{z}.
\]

(47)
Substituting $\xi = z^2 + w^2 - 2zw \cosh \zeta$ into (33), and multiplying both sides by $4z$, we deduce that
\[
\begin{align*}
  u'' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (4z^3 - 8z^2w \cosh \zeta + 4zw^2) \\
+ u' \left(z^2 + w^2 - 2zw \cosh \zeta\right) \cdot 4z \\
= -zu \left(z^2 + w^2 - 2zw \cosh \zeta\right).
\end{align*}
\] (49) (50) (51)

Therefore the integrand in (45)–(46) can be written as follows
\[
\begin{align*}
  u'' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (4z^3 - 8z^2w \cosh \zeta + 4zw^2) \cosh \nu \zeta \\
+ u' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (4z - 2w \cosh \zeta) \cosh \nu \zeta \\
= -zu \left(z^2 + w^2 - 2zw \cosh \zeta\right) \cosh \nu \zeta \\
+ u'' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (4zw^2 \sinh^2 \zeta) \cosh \nu \zeta \\
+ u' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (-2w \cosh \zeta) \cosh \nu \zeta.
\end{align*}
\] (52) (53) (54) (55) (56)

Substituting this into (45), and adding $zy$ to both sides, we obtain
\[
(zy)'(z) + zy(z) =
\]
\[
= \int_C u'' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (4zw^2 \sinh^2 \zeta) \cosh \nu \zeta \, d\zeta
\]
\[
+ \int_C u' \left(z^2 + w^2 - 2zw \cosh \zeta\right) (-2w \cosh \zeta) \cosh \nu \zeta \, d\zeta
\]
\[
+ u'(0) \left(z^2 - w^2\right) \cosh \nu (\log z - \log w) \cdot \frac{1}{z}
\]
\[
+ u(0) \sinh \nu (\log z - \log w) \cdot \frac{\nu}{z}.
\] (57) (58) (59) (60) (61)

In (63), (64), (67), (70), (76), (78), we use integration by parts, and a prime symbol applied to an expression in parentheses $(\ldots)'$ denotes differentiation with respect
to \( \zeta \). First we integrate by parts the integral in (58)

\[
\int_C u'' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( 4zw^2 \sinh^2 \zeta \right) \cosh \nu \zeta \, d\zeta = \tag{62}
\]

\[
= \int_C \left( u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \right)' \left( -2w \sinh \zeta \cosh \nu \zeta \right) \, d\zeta \tag{63}
\]

\[
= -\int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( -2w \sinh \zeta \cosh \nu \zeta \right)' \, d\zeta \tag{64}
\]

\[
+ \left[ u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( -2w \sinh \zeta \cosh \nu \zeta \right) \right]_{\zeta = \log z - \log w} \tag{65}
\]

\[
- \left[ u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( -2w \sinh \zeta \cosh \nu \zeta \right) \right]_{\zeta = 0} \tag{66}
\]

\[
= \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( 2w \sinh \zeta \cosh \nu \zeta \right)' \, d\zeta \tag{67}
\]

\[
- u'(0) \left( z^2 - w^2 \right) \cosh \nu \left( \log z - \log w \right) \cdot \frac{1}{z}. \tag{68}
\]

Substituting this into (58), we obtain

\[
(zy)'(z) + zy(z) = \tag{69}
\]

\[
= \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( 2w \sinh \zeta \cosh \nu \zeta \right)' \, d\zeta \tag{70}
\]

\[
+ \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( -2w \cosh \zeta \cosh \nu \zeta \right) \, d\zeta \tag{71}
\]

\[
+ u(0) \sinh \nu \left( \log z - \log w \right) \cdot \frac{\nu}{z} \tag{72}
\]

\[
= \int_C u' \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( 2w \nu \sinh \zeta \cosh \nu \zeta \right) \, d\zeta \tag{73}
\]

\[
+ u(0) \sinh \nu \left( \log z - \log w \right) \cdot \frac{\nu}{z}. \tag{74}
\]
We conclude the proof by integrating (73) by parts

\[(zy')'(z) + zy(z) = \int_C \left( u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( \frac{-\nu}{z} \sinh \nu \zeta \right) \right) \, d\zeta \] (75)

\[= \int u(0) \sinh \nu (\log z - \log w) \cdot \frac{\nu}{z} \] (77)

\[= -\int u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( \frac{-\nu}{z} \sinh \nu \zeta \right)' \, d\zeta \] (78)

\[+ \left[ u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( \frac{-\nu}{z} \sinh \nu \zeta \right) \right]_{\zeta = \log z - \log w}^{\zeta = 0} \] (79)

\[+ \int u \left( z^2 + w^2 - 2zw \cosh \zeta \right) \left( \frac{\nu^2}{z} \cosh \nu \zeta \right) \, d\zeta \] (82)

\[= \frac{\nu^2}{z} y(z). \] (83)

\[\Box \]

**Lemma 4.2.** If \( \nu \neq 0 \) and \( r > 0 \), then

\[
\lim_{w \to 0^+} \left( J_\nu(w)Y_\nu(rw) - Y_\nu(w)J_\nu(rw) \right) = \frac{1}{\pi \nu} \left( r^\nu - r^{-\nu} \right). \] (84)

If \( r > 0 \), then

\[
\lim_{w \to 0^+} \left( J_0(w)Y_0(rw) - Y_0(w)J_0(rw) \right) = \frac{2}{\pi} \log r. \] (85)

**Proof.** First we assume that \( \nu \) is non-integer. Therefore [3, 8.403(1)]

\[
Y_\nu(z) = \cot \nu \pi J_\nu(z) - \frac{1}{\sin \nu \pi} J_{-\nu}(z) \] (86)

for \( |\text{Arg } z| < \pi \). The lowest order term in the power series expansion of \( J_\nu(z) \) [3, 8.402] equals \( \frac{z^\nu}{\nu \Gamma(\nu + 1)} \), or equivalently \( \frac{z^\nu}{\nu \Gamma(\nu)} \). Consequently,

\[
\lim_{w \to 0^+} \left( J_\nu(w)Y_\nu(rw) - Y_\nu(w)J_\nu(rw) \right) = \] (87)

\[= -\frac{1}{\sin \nu \pi} \lim_{w \to 0^+} \left( J_\nu(w)J_{-\nu}(rw) - J_{-\nu}(w)J_\nu(rw) \right) \] (88)

\[= \frac{1}{(\sin \nu \pi) \nu \Gamma(\nu) \Gamma(1 - \nu)} \left( r^\nu - r^{-\nu} \right) \] (89)

\[= \frac{1}{\pi \nu} \left( r^\nu - r^{-\nu} \right). \] (90)
The last step uses Euler’s reflection formula for the gamma function [13, 5.5.3]. If \( \nu \) is a positive integer, then the lowest order term in the power series expansion of \( Y_\nu(z) \) [3, 8.403(2)] equals \((-1/\pi)^2(\nu-1)!z^{-\nu}\), and (84) follows in a similar fashion. If \( \nu \) is a negative integer, then \( J_\nu = (-1)^\nu J_{-\nu} \) and \( Y_\nu = (-1)^\nu Y_{-\nu} \) [3, 8.404], and (84) follows from the previous case. Finally, for \( \nu = 0 \) we have [3, 8.444(1)]

\[
Y_0(z) = \frac{2}{\pi} J_0(z) \log z + u_0(z),
\]

where \( u_0 \) is an entire function, and (85) follows.

We present below an alternative proof of Theorem 3.1, which uses the Mehler-Sonine integrals and proceeds along the lines of [15, 13.72].

\textbf{Proof.} Let \( z = x + iy \). By assumption, \( y > 0 \) and \( \nu \) is real. We start with the Mehler-Sonine representation [13, 10.9.10]

\[
H^{(1)}_\nu(z) = e^{-\frac{1}{2} \nu \pi i} \int_{-\infty}^{\infty} e^{iz \cosh t - \nu t} dt,
\]

valid when \( \text{Im}(z) > 0 \). Taking the complex conjugates of both sides, we obtain

\[
\overline{H^{(1)}_\nu(z)} = -e^{\frac{1}{2} \nu \pi i} \int_{-\infty}^{\infty} e^{-iz \cosh s - \nu s} ds.
\]

We multiply these equations side by side, and replace the product of the absolutely convergent integrals by a double integral

\[
|H^{(1)}_\nu(z)|^2 = \frac{1}{\pi^2} \int_{\mathbb{R}^2} e^{iz \cosh t - \nu \cosh s} e^{-2y \cosh u \cosh v - 2\nu u} ds dt.
\]

We make the substitution \( u = \frac{s+t}{2} \) and \( v = \frac{s-t}{2} \), and express the argument of the exponential in the new coordinates

\[
iz \cosh t - i\nu \cosh s - \nu(s+t) =
iz(x(\cosh t - \cosh s) - y(\cosh t + \cosh s) - \nu(s + t))
= 2ix \sinh u \sinh v - 2y \cosh u \cosh v - 2\nu u.
\]

Since \( ds dt = 2dudv \), in the new coordinates (94) becomes

\[
|H^{(1)}_\nu(z)|^2 = \frac{2}{\pi^2} \int_{\mathbb{R}^2} e^{2ix \sinh u \sinh v} e^{-2y \cosh u \cosh v - 2\nu u} dudv.
\]
Since $|H^{(1)}_{\nu}(z)|^2$ is real, we can omit the imaginary part of $e^{2x \sinh u \sinh v}$ in (98), which leaves us with $\cos(2x \sinh u \sinh v)$. Next, we write (98) as an iterated integral

$$|H^{(1)}_{\nu}(z)|^2 = \frac{2}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(2x \sinh u \sinh v) \ e^{-2y \cosh u \cosh v} \ dv \ du. \quad (99)$$

The inner integral can be evaluated using the substitution $w = \sinh v$

$$\int_{-\infty}^{\infty} \cos(2x \sinh u \sinh v) \ e^{-2y \cosh u \cosh v} \ dv =$$

$$= \int_{-\infty}^{\infty} \cos(2x \sinh u \ w) \ e^{-2y \cosh \sqrt{1+w^2}} \ \frac{dw}{\sqrt{1+w^2}} \quad (101)$$

$$= 2 \int_{0}^{\infty} \cos(2x \sinh \sqrt{1+w^2} u) \ e^{-2y \cosh \sqrt{1+w^2}} \ \frac{dw}{\sqrt{1+w^2}} \quad (102)$$

$$= 2K_0 \left( 2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u} \right), \quad (103)$$

where the last step uses formula [3, 3.914(4)]. Substituting this into (99), we obtain

$$|H^{(1)}_{\nu}(z)|^2 = \frac{4}{\pi^2} \int_{-\infty}^{\infty} K_0 \left( 2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u} \right) e^{-2\nu u} \ du \quad (104)$$

$$= \frac{8}{\pi^2} \int_{0}^{\infty} K_0 \left( 2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u} \right) \cosh 2\nu u \ du. \quad (105)$$

This completes our alternative proof of (14). \qed

References


