A Nicholson-type integral for the cross-product of the Bessel functions

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Abstract

We prove a new Nicholson-type integral representation for the cross-product of the Bessel functions $J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z)$, and related integral representations for $|H_\nu^{(1)}(z)|^2$, where $H_\nu^{(1)}$ is the Hankel function of the first kind.

Key words: Nicholson's formula, cross-product, Bessel functions, Hankel functions

1. Introduction

The main result of this paper is the following representation of the cross-product of the Bessel functions J_ν and Y_ν

$$J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = \tag{1}$$

$$= \frac{2}{\pi}\int_C J_0\left(\sqrt{z^2 + w^2 - 2zw\cosh\zeta}\right)\cosh\nu\zeta \, d\zeta, \tag{2}$$

where C is any contour starting at 0 and ending at $\log z - \log w$. The formula holds for every complex number ν, and all complex numbers z and w such that $|\operatorname{Arg} z| < \pi$ and $|\operatorname{Arg} w| < \pi$. We demonstrate it in Theorem 2.1 by verifying that the integral satisfies Bessel's differential equation and appropriate limiting conditions at the origin.

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The cross-products of the Bessel functions have long been studied theoretically, see e.g. [13, 10.6, 10.21], [8], [7]. They have also found several applications to wave propagation in cylindrical geometries, e.g. to acoustic modeling of ducts [5], and to modeling of optical waveguides [6], [14]. Other applications include heat transport [2] and free elastic oscillations of a spherical body [10]. Other integral representations of various products of the Bessel functions are well-known, an extensive list is given in [9, p. 93-98], see also [11].

Our second result is an integral representation of the square of the modulus of the Hankel function $H_\nu^{(1)}$ of the first kind

$$|H_\nu^{(1)}(x+\imath y)|^2 = \frac{8}{\pi^2}\int_0^\infty K_0\left(2\sqrt{x^2\sinh^2 t + y^2\cosh^2 t}\right)\cosh 2\nu t\, dt. \qquad (3)$$

This formula is valid if all x, y, ν are real, and $y > 0$. We present two simple derivations: one in Theorem 3.1 is based on a known representation of the product of the modified Bessel functions $K_\nu(\alpha)K_\nu(\beta)$, and another one in Appendix uses the Mehler-Sonine integrals.

Formula (3) has some interesting consequences. For example, it follows from (3) that for a fixed z in the upper half-plane, the magnitude $|H_\nu^{(1)}(z)|$ increases with the order ν when $\nu \geqslant 0$. This fact has been observed experimentally [4], [1], but no proofs are given there. The property is needed in the numerical evaluation of the Bessel and the Hankel functions using a three-term recurrence. For the sake of numerical stability, the recurrence must proceed in the direction of increasing magnitudes. It is known that the magnitudes of the modified Bessel function $|K_\nu(\zeta)|$ grow with $\nu \geqslant 0$ [13, 10.37] when ζ is in the right half-plane, and monotonicity of $|H_\nu^{(1)}(z)|$ can also be deduced from this fact.

In 1910, J. W. Nicholson published [12] the following equality

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\,\cosh 2\nu t\; dt, \qquad (4)$$

valid for $\mathrm{Re}(z) > 0$ and an arbitrary complex order ν. Nicholson's formula can be recovered from (3) by fixing $x > 0$, letting $y \to 0^+$, and then taking analytic continuation first in z, and then in ν. This derivation of (4) appears somewhat simpler than the one presented in [15], while another short proof is given in [16].

Finally, in Theorem 3.2 we combine (1) and (3) to obtain the following formula

$$\begin{aligned} |H_\nu^{(1)}(x-\imath y)|^2 &= \frac{8}{\pi^2}\int_0^\infty K_0\left(2\sqrt{x^2\sinh^2 t + y^2\cosh^2 t}\right)\cosh 2\nu t\; dt \\ &+ \frac{8}{\pi}\int_0^{\operatorname{arccot}\frac{x}{y}} I_0\left(2\sqrt{-x^2\sin^2\phi + y^2\cos^2\phi}\right)\cos 2\nu\phi\; d\phi, \end{aligned} \qquad (5)$$

which is valid if all x, y, ν are real, and $y > 0$.

2. Integral representation of the cross-product

The following theorem, which gives an integral representation of the cross-product of the Bessel functions J_ν and Y_ν, is the main result of this paper.

Theorem 2.1. *For all complex numbers z and w such that $|\operatorname{Arg} z| < \pi$ and $|\operatorname{Arg} w| < \pi$, and for every complex number ν,*

$$J_\nu(w)Y_\nu(z) - Y_\nu(w)J_\nu(z) = \tag{6}$$

$$= \frac{2}{\pi}\int_C J_0\left(\sqrt{z^2+w^2-2zw\cosh\zeta}\right)\cosh\nu\zeta\, d\zeta, \tag{7}$$

where C is any contour starting at 0 and ending at $\log z - \log w$.

In this paper, J_ν, Y_ν and log always denote the principal branches of these functions. $\operatorname{Arg} z$ denotes the principal argument of a complex number $z \neq 0$. The function $J_0(\sqrt{\cdot}\,)$ is entire, so the integral does not depend on the choice of a contour joining 0 to $\log z - \log w$.

Proof. Let $\alpha(z,w)$ denote the expression in (7), i.e.

$$\alpha(z,w) = \frac{2}{\pi}\int_C J_0\left(\sqrt{z^2+w^2-2zw\cosh\zeta}\right)\cosh\nu\zeta\, d\zeta. \tag{8}$$

Since the integrand in (8) is symmetric with respect to z and w, and the contour joins 0 to $\log z - \log w$, we infer that

$$\alpha(w,z) = -\alpha(z,w). \tag{9}$$

It is shown in Lemma 4.1 (see Appendix) that for a fixed $w \neq 0$, $\alpha(z,w)$ satisfies Bessel's equation with respect to z in the region $|\operatorname{Arg} z| < \pi$. Since J_ν and Y_ν are linearly independent solutions of Bessel's equation [3, 8.474], we have

$$\alpha(z,w) = c_1(w)J_\nu(z) + c_2(w)Y_\nu(z), \tag{10}$$

for some functions c_1 and c_2. Combining this with (9), we obtain

$$c_1(z)J_\nu(w) + c_2(z)Y_\nu(w) = -c_1(w)J_\nu(z) - c_2(w)Y_\nu(z). \tag{11}$$

Let w_1 and w_2 be any complex numbers such that $J_\nu(w_1)Y_\nu(w_2) - Y_\nu(w_1)J_\nu(w_2) \neq 0$. Setting $w = w_1$ and $w = w_2$ in (11), we obtain a system of two linear equations for $c_1(z)$ and $c_2(z)$. Therefore both $c_1(z)$ and $c_2(z)$ are linear combinations of $J_\nu(z)$ and $Y_\nu(z)$ with coefficients that do not depend on z or w. Consequently, (10) implies that

$$\alpha(z,w) = a_{11}J_\nu(w)J_\nu(z) + a_{21}Y_\nu(w)J_\nu(z) + a_{12}J_\nu(w)Y_\nu(z) + a_{22}Y_\nu(w)Y_\nu(z), \tag{12}$$

where the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ do not depend on z or w, but may depend on ν. Due to linear independence of J_ν and Y_ν, we infer from (9) that $a_{11} = a_{22} = 0$, and $a_{21} = -a_{12}$.

We determine the remaining coefficient a_{12} by comparing the limits of (6) and (7) as z and w approach 0 through the positive real values with $\frac{z}{w} = r$, where $r > 0$. From (8), we obtain for $\nu \neq 0$

$$\lim_{w\to 0^+} \alpha(rw, w) = \frac{2}{\pi} J_0(0) \int_0^{\log r} \cosh \nu\zeta \, d\zeta = \frac{1}{\pi\nu} \left(r^\nu - r^{-\nu}\right). \tag{13}$$

For $\nu = 0$, the above limit equals $\frac{2}{\pi} \log r$. The same values are obtained in Lemma 4.2 (see Appendix) for the corresponding limit of the cross-product. Therefore $a_{12} = 1$, and the proof is complete. □

3. Integral representations of $|H_\nu^{(1)}(z)|^2$

In this section, we assume that the order ν is real. We consider the cases when $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$ separately.

Theorem 3.1. *If x is a real number and $y > 0$, then*

$$|H_\nu^{(1)}(x + \imath y)|^2 = \frac{8}{\pi^2} \int_0^\infty K_0 \left(2\sqrt{x^2 \sinh^2 t + y^2 \cosh^2 t}\right) \cosh 2\nu t \, dt, \tag{14}$$

for every real order ν.

K_ν denotes the modified Bessel function of the second kind and order ν, and $H_\nu^{(1)}$ denotes the Hankel function of the first kind and order ν.

Proof. The following formula, valid when $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, is a special case of [3, 6.648] (with ρ replaced by ν, and x replaced by s)

$$K_\nu(\alpha)K_{-\nu}(\beta) = \frac{1}{2} \int_{-\infty}^\infty e^{\nu s} K_0 \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh s}\right) ds. \tag{15}$$

Since $K_\nu = K_{-\nu}$ [13, 10.27.3], we have

$$K_\nu(\alpha)K_\nu(\beta) = \frac{1}{2} \int_{-\infty}^\infty K_0 \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh s}\right) \cosh \nu s \, ds \tag{16}$$

$$= \int_0^\infty K_0 \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh s}\right) \cosh \nu s \, ds \tag{17}$$

$$= 2 \int_0^\infty K_0 \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh 2t}\right) \cosh 2\nu t \, dt. \tag{18}$$

Since ν is real, we have [13, 10.11.9]

$$\overline{H_\nu^{(1)}(z)} = H_\nu^{(2)}(\overline{z}). \tag{19}$$

From connection formulas between K_ν and $H_\nu^{(1)}$, $H_\nu^{(2)}$ [13, 10.27.8], we infer that

$$H_\nu^{(1)}(z)H_\nu^{(2)}(w) = \frac{4}{\pi^2}\,K_\nu(-\imath z)K_\nu(\imath w), \tag{20}$$

when $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) < 0$. Combining (19) and (20) (with $w = \overline{z}$), we obtain

$$|H_\nu^{(1)}(z)|^2 = H_\nu^{(1)}(z)H_\nu^{(2)}(\overline{z}) = \frac{4}{\pi^2}\,K_\nu(-\imath z)K_\nu(\imath \overline{z}). \tag{21}$$

The claim now follows by setting $z = x + \imath y$, $\alpha = -\imath z$, $\beta = \imath\overline{z}$ in (18). $\square$

Another, more direct proof of this theorem is given in Appendix.

Theorem 3.2. *If x is a real number and $y > 0$, then*

$$|H_\nu^{(1)}(x - \imath y)|^2 = \frac{8}{\pi^2}\int_0^\infty K_0\left(2\sqrt{x^2\sinh^2 t + y^2\cosh^2 t}\right)\cosh 2\nu t\; dt \tag{22}$$

$$+\;\frac{8}{\pi}\int_0^{\operatorname{arccot}\frac{x}{y}} I_0\left(2\sqrt{-x^2\sin^2\phi + y^2\cos^2\phi}\right)\cos 2\nu\phi\; d\phi \tag{23}$$

for every real order ν.

I_0 denotes the modified Bessel function of the first kind and order 0, and arccot denotes the inverse function of the cotangent $\cot\phi, 0 < \phi < \pi$.

Proof. Replacing z with $\overline{z}$ in (19), we obtain

$$\overline{H_\nu^{(1)}(\overline{z})} = H_\nu^{(2)}(z). \tag{24}$$

Combining this with (21) and connection formulas between the Hankel and the Bessel functions [13, 10.4.3], we have

$$|H_\nu^{(1)}(\overline{z})|^2 = H_\nu^{(1)}(\overline{z})H_\nu^{(2)}(z) - H_\nu^{(1)}(z)H_\nu^{(2)}(\overline{z}) + |H_\nu^{(1)}(z)|^2 \tag{25}$$

$$= -2\imath\left(J_\nu(\overline{z})Y_\nu(z) - Y_\nu(\overline{z})J_\nu(z)\right) + |H_\nu^{(1)}(z)|^2. \tag{26}$$

We set $z = x + \imath y$, $w = x - \imath y$, and express the above cross-product by means of Theorem 2.1. Choosing the contour C to be the line interval joining 0 to

$\log z - \log w = 2\imath \operatorname{arccot} \frac{x}{y}$, we obtain

$$J_\nu(\overline{z})Y_\nu(z) - Y_\nu(\overline{z})J_\nu(z) = \tag{27}$$
$$= \frac{2}{\pi} \int_0^{2 \operatorname{arccot} \frac{x}{y}} J_0\left(\sqrt{z^2 + w^2 - 2zw\cosh(\imath t)}\right) \cosh(\nu \imath t)\, \imath\, dt \tag{28}$$
$$= \frac{4\imath}{\pi} \int_0^{\operatorname{arccot} \frac{x}{y}} J_0\left(\sqrt{2x^2 - 2y^2 - 2(x^2+y^2)\cos(2\phi)}\right) \cos(2\nu\phi)\, d\phi \tag{29}$$
$$= \frac{4\imath}{\pi} \int_0^{\operatorname{arccot} \frac{x}{y}} J_0\left(2\sqrt{x^2 \sin^2\phi - y^2\cos^2\phi}\right) \cos 2\nu\phi\, d\phi. \tag{30}$$

The claim now follows by substituting the above and (14) into (26), and using a connection formula between J_0 and I_0 [13, 10.27.6]. □

Since $|H_\nu^{(2)}(z)| = |H_\nu^{(1)}(\overline{z})|$, see (24), both theorems have their counterparts for $|H_\nu^{(2)}(z)|^2$.

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4. Appendix

The following two lemmas are used in the proof of Theorem 2.1.

Lemma 4.1. *For a fixed $w \neq 0$, the function*

$$y(z) = \int_C J_0\left(\sqrt{z^2 + w^2 - 2zw\cosh\zeta}\right) \cosh\nu\zeta\, d\zeta, \tag{31}$$

where C is any contour joining 0 to $\log z - \log w$, satisfies Bessel's differential equation

$$(zy')' + \left(z - \frac{\nu^2}{z}\right) y = 0 \tag{32}$$

in the region $|\operatorname{Arg} z| < \pi$.

Proof. We introduce the function $u(\xi) = J_0(\sqrt{\xi})$, which is entire, and satisfies the following differential equation [3, 8.491(9)]

$$\xi u'' + u' + \frac{1}{4}u = 0. \tag{33}$$

We write the contour integral (31) as follows

$$y(z) = \int_C u\left(z^2 + w^2 - 2zw\cosh\zeta\right)\, \cosh\nu\zeta\ d\zeta. \tag{34}$$

Differentiating this formula under the integral sign with respect to z, we obtain

$$\begin{aligned} y'(z) &= \int_C u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(2z - 2w\cosh\zeta)\, \cosh\nu\zeta\ d\zeta & (35)\\ &+ \left[u\left(z^2 + w^2 - 2zw\cosh\zeta\right)\cosh\nu\zeta\right]_{\zeta=\log z - \log w}(\log z - \log w)' \\ &= \int_C u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(2z - 2w\cosh\zeta)\, \cosh\nu\zeta\ d\zeta & (36)\\ &+ u(0)\cosh\nu(\log z - \log w)\cdot\frac{1}{z}. & (37)\end{aligned}$$

Consequently,

$$\begin{aligned} zy'(z) &= \int_C u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(2z^2 - 2zw\cosh\zeta\right)\, \cosh\nu\zeta\ d\zeta & (38)\\ &+ u(0)\cosh\nu(\log z - \log w). & (39)\end{aligned}$$

Differentiating again, we obtain

$$\begin{aligned} &(zy')'(z) = & (40)\\ &= \int_C \Big[u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)(2z - 2w\cosh\zeta)\left(2z^2 - 2zw\cosh\zeta\right)\, \cosh\nu\zeta & (41)\\ &\quad + u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(4z - 2w\cosh\zeta)\cosh\nu\zeta\Big]d\zeta & (42)\\ &+ \left[u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(2z^2 - 2zw\cosh\zeta\right)\, \cosh\nu\zeta\right]_{\zeta=\log z - \log w}(\log z - \log w)' \\ &+ u(0)\sinh\nu(\log z - \log w)\cdot\frac{\nu}{z}. & (43)\end{aligned}$$

Collecting the terms, we arrive at

$$\begin{aligned} &(zy')'(z) = & (44)\\ &= \int_C \Big[u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(4z^3 - 8z^2 w\cosh\zeta + 4zw^2\cosh^2\zeta\right)\, \cosh\nu\zeta & (45)\\ &\quad + u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(4z - 2w\cosh\zeta)\cosh\nu\zeta\Big]d\zeta & (46)\\ &+ u'(0)\left(z^2 - w^2\right)\cosh\nu(\log z - \log w)\cdot\frac{1}{z} & (47)\\ &+ u(0)\sinh\nu(\log z - \log w)\cdot\frac{\nu}{z}. & (48)\end{aligned}$$

Substituting $\xi = z^2 + w^2 - 2zw\cosh\zeta$ into (33), and multiplying both sides by $4z$, we deduce that

$$u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(4z^3 - 8z^2w\cosh\zeta + 4zw^2\right) \tag{49}$$
$$+\; u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)\cdot 4z \tag{50}$$
$$= \;-zu\left(z^2 + w^2 - 2zw\cosh\zeta\right). \tag{51}$$

Therefore the integrand in (45)–(46) can be written as follows

$$u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(4z^3 - 8z^2w\cosh\zeta + 4zw^2\cosh^2\zeta\right)\,\cosh\nu\zeta \tag{52}$$
$$+\; u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(4z - 2w\cosh\zeta)\,\cosh\nu\zeta \tag{53}$$
$$= \;-zu\left(z^2 + w^2 - 2zw\cosh\zeta\right)\,\cosh\nu\zeta \tag{54}$$
$$+\; u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(4zw^2\sinh^2\zeta\right)\,\cosh\nu\zeta \tag{55}$$
$$+\; u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(-2w\cosh\zeta)\,\cosh\nu\zeta. \tag{56}$$

Substituting this into (45), and adding zy to both sides, we obtain

$$(zy')'(z) + zy(z) = \tag{57}$$
$$= \int_C u''\left(z^2 + w^2 - 2zw\cosh\zeta\right)\left(4zw^2\sinh^2\zeta\right)\,\cosh\nu\zeta\; d\zeta \tag{58}$$
$$+ \int_C u'\left(z^2 + w^2 - 2zw\cosh\zeta\right)(-2w\cosh\zeta)\,\cosh\nu\zeta\; d\zeta \tag{59}$$
$$+\; u'(0)\left(z^2 - w^2\right)\cosh\nu(\log z - \log w)\cdot\frac{1}{z} \tag{60}$$
$$+\; u(0)\sinh\nu(\log z - \log w)\cdot\frac{\nu}{z}. \tag{61}$$

In (63), (64), (67), (70), (76), (78), we use integration by parts, and a prime symbol applied to an expression in parentheses $(\ldots)'$ denotes differentiation with respect to ζ. First we integrate by parts the integral in (58)

$$\int_C u''\left(z^2+w^2-2zw\cosh\zeta\right)\left(4zw^2\sinh^2\zeta\right)\,\cosh\nu\zeta\,d\zeta = \tag{62}$$

$$= \int_C \left(u'\left(z^2+w^2-2zw\cosh\zeta\right)\right)'\left(-2w\sinh\zeta\,\cosh\nu\zeta\right)\,d\zeta \tag{63}$$

$$= -\int_C u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(-2w\sinh\zeta\,\cosh\nu\zeta\right)'\,d\zeta \tag{64}$$

$$+ \left[u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(-2w\sinh\zeta\,\cosh\nu\zeta\right)\right]_{\zeta=\log z-\log w} \tag{65}$$

$$- \left[u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(-2w\sinh\zeta\,\cosh\nu\zeta\right)\right]_{\zeta=0} \tag{66}$$

$$= \int_C u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(2w\sinh\zeta\,\cosh\nu\zeta\right)'\,d\zeta \tag{67}$$

$$- u'(0)\left(z^2-w^2\right)\cosh\nu(\log z-\log w)\cdot\frac{1}{z}. \tag{68}$$

Substituting this into (58), we obtain

$$(zy')'(z)+zy(z) = \tag{69}$$

$$= \int_C u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(2w\sinh\zeta\,\cosh\nu\zeta\right)'\,d\zeta \tag{70}$$

$$+ \int_C u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(-2w\cosh\zeta\,\cosh\nu\zeta\right)\,d\zeta \tag{71}$$

$$+ u(0)\sinh\nu(\log z-\log w)\cdot\frac{\nu}{z} \tag{72}$$

$$= \int_C u'\left(z^2+w^2-2zw\cosh\zeta\right)\left(2w\nu\sinh\zeta\,\sinh\nu\zeta\right)\,d\zeta \tag{73}$$

$$+ u(0)\sinh\nu(\log z-\log w)\cdot\frac{\nu}{z}. \tag{74}$$

We conclude the proof by integrating (73) by parts

$$(zy')'(z) + zy(z) = \tag{75}$$

$$= \int_C \left(u\left(z^2+w^2-2zw\cosh\zeta\right)\right)' \left(-\frac{\nu}{z}\,\sinh\nu\zeta\right)\,d\zeta \tag{76}$$

$$+\;\; u(0)\sinh\nu(\log z-\log w)\cdot\frac{\nu}{z} \tag{77}$$

$$= -\int_C u\left(z^2+w^2-2zw\cosh\zeta\right)\left(-\frac{\nu}{z}\,\sinh\nu\zeta\right)'\,d\zeta \tag{78}$$

$$+\;\; \left[u\left(z^2+w^2-2zw\cosh\zeta\right)(-\frac{\nu}{z}\,\sinh\nu\zeta)\right]_{\zeta=\log z-\log w} \tag{79}$$

$$-\;\; \left[u\left(z^2+w^2-2zw\cosh\zeta\right)(-\frac{\nu}{z}\,\sinh\nu\zeta)\right]_{\zeta=0} \tag{80}$$

$$+\;\; u(0)\sinh\nu(\log z-\log w)\cdot\frac{\nu}{z} \tag{81}$$

$$= \int_C u\left(z^2+w^2-2zw\cosh\zeta\right)\left(\frac{\nu^2}{z}\,\cosh\nu\zeta\right)\,d\zeta \tag{82}$$

$$= \frac{\nu^2}{z}\,y(z). \tag{83}$$

□

Lemma 4.2. *If $\nu \neq 0$ and $r > 0$, then*

$$\lim_{w\to 0^+}\left(J_\nu(w)Y_\nu(rw) - Y_\nu(w)J_\nu(rw)\right) = \frac{1}{\pi\nu}\left(r^\nu - r^{-\nu}\right). \tag{84}$$

If $r > 0$, then

$$\lim_{w\to 0^+}\left(J_0(w)Y_0(rw) - Y_0(w)J_0(rw)\right) = \frac{2}{\pi}\log r. \tag{85}$$

Proof. First we assume that ν is non-integer. Therefore [3, 8.403(1)]

$$Y_\nu(z) = \cot\nu\pi\; J_\nu(z) - \frac{1}{\sin\nu\pi}\;J_{-\nu}(z) \tag{86}$$

for $|\operatorname{Arg} z| < \pi$. The lowest order term in the power series expansion of $J_\nu(z)$ [3, 8.402] equals $\frac{z^\nu}{2^\nu\Gamma(\nu+1)}$, or equivalently $\frac{z^\nu}{2^\nu\nu\Gamma(\nu)}$. Consequently,

$$\lim_{w\to 0^+}\left(J_\nu(w)Y_\nu(rw) - Y_\nu(w)J_\nu(rw)\right) = \tag{87}$$

$$= -\frac{1}{\sin\nu\pi}\lim_{w\to 0^+}\left(J_\nu(w)J_{-\nu}(rw) - J_{-\nu}(w)J_\nu(rw)\right) \tag{88}$$

$$= \frac{1}{(\sin\nu\pi)\,\nu\,\Gamma(\nu)\,\Gamma(1-\nu)}\left(r^\nu - r^{-\nu}\right) \tag{89}$$

$$= \frac{1}{\pi\nu}\left(r^\nu - r^{-\nu}\right). \tag{90}$$

The last step uses Euler's reflection formula for the gamma function [13, 5.5.3]. If ν is a positive integer, then the lowest order term in the power series expansion of $Y_\nu(z)$ [3, 8.403(2)] equals $(-1/\pi)2^\nu(\nu-1)!z^{-\nu}$, and (84) follows in a similar fashion. If ν is a negative integer, then $J_\nu = (-1)^\nu J_{-\nu}$ and $Y_\nu = (-1)^\nu Y_{-\nu}$ [3, 8.404], and (84) follows from the previous case. Finally, for $\nu = 0$ we have [3, 8.444(1)]

$$Y_0(z) = \frac{2}{\pi}\, J_0(z) \log z + u_0(z), \tag{91}$$

where u_0 is an entire function, and (85) follows. □

We present below an alternative proof of Theorem 3.1, which uses the Mehler-Sonine integrals and proceeds along the lines of [15, 13.72].

Proof. Let $z = x + \imath y$. By assumption, $y > 0$ and ν is real. We start with the Mehler-Sonine representation [13, 10.9.10]

$$H_\nu^{(1)}(z) = \frac{e^{-\frac{1}{2}\nu\pi\imath}}{\pi\imath} \int_{-\infty}^{\infty} e^{\imath z \cosh t - \nu t}\, dt, \tag{92}$$

valid when $\mathrm{Im}(z) > 0$. Taking the complex conjugates of both sides, we obtain

$$\overline{H_\nu^{(1)}(z)} = -\frac{e^{\frac{1}{2}\nu\pi\imath}}{\pi\imath} \int_{-\infty}^{\infty} e^{-\imath \bar{z} \cosh s - \nu s}\, ds. \tag{93}$$

We multiply these equations side by side, and replace the product of the absolutely convergent integrals by a double integral

$$|H_\nu^{(1)}(z)|^2 = \frac{1}{\pi^2} \iint_{\mathbb{R}^2} e^{\imath z \cosh t - \imath \bar{z} \cosh s - \nu(s+t)}\, dsdt. \tag{94}$$

We make the substitution $u = \frac{s+t}{2}$ and $v = \frac{-s+t}{2}$, and express the argument of the exponential in the new coordinates

$$\begin{aligned} \imath z \cosh t - \imath \bar{z} \cosh s - \nu(s+t) = & \qquad (95)\\ = \imath x(\cosh t - \cosh s) - y(\cosh t + \cosh s) - \nu(s+t) & \qquad (96)\\ = 2\imath x \sinh u \sinh v - 2y \cosh u \cosh v - 2\nu u. & \qquad (97) \end{aligned}$$

Since $dsdt = 2dudv$, in the new coordinates (94) becomes

$$|H_\nu^{(1)}(z)|^2 = \frac{2}{\pi^2} \iint_{\mathbb{R}^2} e^{2\imath x \sinh u \sinh v}\; e^{-2y \cosh u \cosh v - 2\nu u}\, dudv. \tag{98}$$

Since $|H_\nu^{(1)}(z)|^2$ is real, we can omit the imaginary part of $e^{2\imath x \sinh u \sinh v}$ in (98), which leaves us with $\cos(2x \sinh u \sinh v)$. Next, we write (98) as an iterated integral

$$|H_\nu^{(1)}(z)|^2 = \frac{2}{\pi^2} \int_{-\infty}^{\infty} e^{-2\nu u} \int_{-\infty}^{\infty} \cos(2x \sinh u \sinh v)\; e^{-2y \cosh u \cosh v}\, dv\, du. \tag{99}$$

The inner integral can be evaluated using the substitution $w = \sinh v$

$$\int_{-\infty}^{\infty} \cos(2x \sinh u \sinh v)\; e^{-2y \cosh u \cosh v}\, dv = \tag{100}$$

$$= \int_{-\infty}^{\infty} \cos(2x\, \text{sinh}u\; w)\; e^{-2y\, \text{cosh}u\, \sqrt{1+w^2}}\; \frac{dw}{\sqrt{1+w^2}} \tag{101}$$

$$= 2\int_{0}^{\infty} \cos(2x\, \text{sinh}u\; w)\; e^{-2y\, \text{cosh}u\, \sqrt{1+w^2}}\; \frac{dw}{\sqrt{1+w^2}} \tag{102}$$

$$= 2K_0\left(2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u}\right), \tag{103}$$

where the last step uses formula [3, 3.914(4)]. Substituting this into (99), we obtain

$$|H_\nu^{(1)}(z)|^2 = \frac{4}{\pi^2} \int_{-\infty}^{\infty} K_0\left(2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u}\right) e^{-2\nu u}\, du \tag{104}$$

$$= \frac{8}{\pi^2} \int_{0}^{\infty} K_0\left(2\sqrt{x^2 \sinh^2 u + y^2 \cosh^2 u}\right) \cosh 2\nu u\, du. \tag{105}$$

This completes our alternative proof of (14). □

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