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A stochastic 2D-model for calculating vibrations in random layers

Peter Balazs, Wolfgang Kreuzer*, Holger Waubke
Austrian Academy of Sciences, Acoustic Research Institute, Reichsratsstrasse 17,A-1010 Vienna, Austria

Vibrations induced by machinery and traffic have become of increasing concern in the last years, for example, when constructing buildings near railway lines. In this paper we will present a model designed to predict the vibration level in the ground. Since in practice it is nearly impossible to determine exact material parameters for soil layers, we use a model with a stochastic shear modulus G. Under moderate assumptions G can be split with the Karhunen-Loeve expansion into a mean value G_0 and a stochastic part G_{stoch}. Using a combination method of finite elements, Fourier transformation and Polynomial Chaos, it is possible to transform the partial differential equation describing the system into a matrix-vector formulation $\boldsymbol{K}\boldsymbol{x} = \boldsymbol{b}$ which can be split into a deterministic and a stochastic part $(\boldsymbol{K}_0 + \boldsymbol{K}_s)\boldsymbol{x} = \boldsymbol{b}$ analog to the shear modulus. To keep the dimensions of the matrices involved with this system small, we use a Neumann-like iteration to solve it. Finally, results for a small example are presented.

Keywords: Vibrations; soil layers; stochastic finite elements

1. Introduction

Ground vibrations decrease rapidly with increasing distance which is one of the reasons they have been neglected in the past. But with increasing traffic, strategies to deal with these annoying vibrations become more and more important, thus making it necessary to develop mathematical models to simulate propagation of vibration in soil.

Up to now consideration was largely given to deterministic models or stochastic models based on perturbation theory[3,2,7,14,15,19,8,6,1,9]. As pointed out by Ghanem and Spanos[12] such models are only applicable to small disturbances of the mean value; thus they neglect a randomness in the solution resulting from a large randomness in the material parameters. But in practice, especially when looking at soil layers, material parameters can have rather large variances. As a way to deal with such problems the concept of stochastic finite elements[12] was introduced.

In earlier publications Waubke[21,22,23,13,20] introduced ideas for simulating vibrations in liquid and solids, which were used to create a numeric 2D finite element model. The randomness in the material parameters is introduced via a shear modulus G that is not only dependent on the depth of the soil layer but also on a random variable θ.

We use the *Karhunen-Loeve expansion* (KLE) to split the shear modulus G into a mean value part G_0, which is independent of θ, and a stochastic part G_{stoch}. The unknown deformations u and w in the spatial x- and z-directions, which are also dependent on the

*Corresponding author, *kreiza@kfs.oeaw.ac.at*

stochastic parameter θ, are expanded into a basis of Hermite polynomials (dependent on θ) and finite element basis functions. After variation with respect to the coefficients of this expansion and calculation of the expectation value on both sides of the equation, the partial differential equation can be transformed into linear system of equations $\boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}_{\text{Ext}}$. Analog to the shear modulus, it is possible to split $\boldsymbol{K}$ into a (deterministic) mean value part $\boldsymbol{K}_0$, which has block tridiagonal structure, and a full matrix $\boldsymbol{K}_s$. Both matrices have very large dimensions, so instead of solving this system directly, a Neumann-like iteration $\boldsymbol{K}_0\boldsymbol{x}_n = \boldsymbol{f}_{\text{Ext}} - \boldsymbol{K}_s\boldsymbol{x}_{n-1}$ is used. This allows us to exploit the special structure of $\boldsymbol{K}_0$ to decouple the system into smaller subsystems which can be solved quite efficiently.

In this paper we present aspects of this model and also give some results for a small example problem. In Sec. 2 we give some background of the mathematics used, present the differential equation describing the system and give a short introduction to the mathematical tools (KLE, Fourier Transformation and Polynomial Chaos) used. Sec. 3,4 and 5 will deal on how to transform the differential equation into a linear system (finite element ansatz and variation) and how the deterministic and the stochastic parts of the system can be treated differently. Based on this we show in Sec. 6 how we can exploit the special structure of $\boldsymbol{K}_0$ to solve the linear system of equations efficiently using an iteration.

Finally, we present some results of a small test example and give a brief overview of future work.

2. Mathematical Background

2.1. *Formulation of the Problem*

The dynamics of the system of wave propagation in soil is described by a (weak) variational formulation[a]

$$\begin{aligned}
&\delta \int_t \int_x L \,\mathrm{d}x\mathrm{d}t = 0 \,, \\
&L = \int_z \frac{1}{2}\left[\left(\frac{2(1-\nu)}{1-2\nu}G(x,z,\theta)\right)(u_x^2 + w_z^2) + \frac{4\nu}{1-2\nu}G(x,z,\theta)u_x w_z + \right. \\
&\qquad \left. + G(x,z,\theta)(u_z + w_x)^2\right] - \frac{1}{2}\rho(u_t^2 + w_t^2)dz - f_{\text{Ext}}w_{|z=0} \,. \qquad (1)
\end{aligned}$$

The unknowns $u := u(x,z,t,\theta)$ and $w := w(x,z,t,\theta)$, which are functions of the spatial variables x and z, the time t and the stochastic variable θ, represent the deformations in the x- and z-directions. The material parameters describing the soil layers are given by the Poisson ratio ν, the density ρ and the shear modulus $G(x,z,\theta)$. The expression x_y is defined as $x_y := \frac{\partial x}{\partial y}$. In the current state we only consider external forces f_{Ext} acting on top (at z=0) in the z-direction.

[a]Here and throughout the whole paper δ is used as the variational symbol unless stated otherwise.

2.2. *Karhunen-Loeve Expansion*

The stochastic shear modulus $G(x, z, \theta)$ is assumed to be a second order random process with mean value $G_0(z)$ and a bounded variance, thus it is possible to use the *Karhunen-Loeve expansion*[12,17,18] (KLE) to split G into a mean value part and a stochastic part:

$$G(x, z, \theta) = G_0(z) + G_s \sum_{i=1}^{n} \sqrt{\lambda_i} f_i(x) \xi_i(\theta) \; . \tag{2}$$

We impose the additional constraint that the mean value $G_0(z)$ of $G(x, z, \theta)$ is also independent of x which will make it easier to decouple the mean value part afterwards.

The (λ_i, f_i) are the solutions of the Fredholm integral equation of the 2nd order

$$\int_D C(x, y) f_i(y) \mathrm{d}y = \lambda_i f_i(x) \; . \tag{3}$$

The $\xi_i(\theta)$ are assumed to be random variables with Gaussian distribution, $G_s C(x, y)$ is the covariance function which is real (except for the complex constant G_s)[b], symmetric and positive definite by definition, furthermore it is assumed that it is independent of the depth z. The eigenvalues of $C(x, y)$ are real and the eigenfunctions form an orthonormal set, i.e.

$$\langle f_i, f_j \rangle := \int_D f_i(x) f_j(x) \mathrm{d}x = \begin{cases} 1, \; i = j \\ 0, \; i \neq j. \end{cases} \tag{4}$$

An analytical solution of Eq. (3) is possible for a few covariance functions, however, for most problems a numerical approach will be necessary. For the time being we restrict ourselves to a very simple approach in order to calculate the eigenpairs (some other ways are described by Ghanem and Spanos[12] or Phoon[18]): The eigenfunctions $f_i(x)$ are approximated on an equidistant grid with stepsize h and the integral in Eq. (3) will be solved using a simple quadrature scheme:

$$\lambda_i f_i(x_k) = \int_{-\infty}^{\infty} C(x_k, y) f_i(y) \mathrm{d}y \approx h \sum_{\ell=-N}^{N} C(x_k, y_\ell) f_i(y_\ell), \quad k = -N, \ldots, N. \tag{5}$$

Thus the problem of finding a solution of Eq. (3) is transformed into an eigenvalue problem $\boldsymbol{C}\boldsymbol{f}_i = \tilde{\lambda}_i \boldsymbol{f}_i$ with $\boldsymbol{C}_{ij} := C(x_i, y_j)$, $\boldsymbol{f}_i := (f_i(y_{-N}) \ldots, f_i(y_N))^T$. In order to get a good approximation for the eigenvalue λ_i, the discrete eigenvalue $\tilde{\lambda}_i$ must be rescaled with the stepsize of the quadrature scheme. Also it is necessary to rescale the numerically computed eigenvectors $\boldsymbol{f}_i$ to satisfy $\langle f_i, f_i \rangle = 1$.

The quality of the approximation can be increased by using a better quadrature scheme (for example Gauss quadrature), but one should keep in mind that with this simple scheme the following Fourier transformation is limited to the same grid that has been used for constructing the eigenvalue problem.

[b]The mean value and the variance of the shear modulus can be complex valued.

2.3. *Polynomial Chaos*

Polynomial Chaos was introduced by Wiener[24] to expand a real second order random process into an expansion in terms of Hermite polynomials. Following Ghanem and Spanos[12] a random process $\mu(\theta)$ can be expanded into

$$\begin{aligned}\mu(\theta) = a_0 H_0 + \sum_{i_1=1}^{\infty} a_{i_1} H_1(\xi_{i1}(\theta)) + \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2} H_2(\xi_{i_2}(\theta), \xi_{i_2}(\theta)) + \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3} H_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) + \dots \ . \end{aligned} \tag{6}$$

The H_i are Hermite polynomials of order i, the ξ_j are orthonormal Gaussian random variables. To simplify notation we rewrite Eq. (6) in the form

$$\mu(\theta) = \sum_i a_i \Gamma_i(\boldsymbol{\xi}), \tag{7}$$

where there is a one-to-one correspondence of the Hermite polynomials H and the functions Γ as well as for the coefficients a_j and $a_{i_1 \dots i_r}$; $\boldsymbol{\xi}$ is defined as the set of the random functions $\xi_i(\theta)$. Using the properties of the Hermite polynomials[12] it is clear that

$$\langle \Gamma_i, \Gamma_j \rangle = \mathbb{E}(\Gamma_i \Gamma_j) = 0 \text{ for } i \neq j. \tag{8}$$

2.4. *Fourier Transformation*

The concepts of the KLE and Polynomial Chaos have been used in the literature to some degree, in our new approach we additionally combine these concepts with a Fourier Transformation and a weak formulation of the differential equation, which makes it possible to reduce the dimension of the system massivly.

In this section we would like to review the definition of the Fourier transformation and point out some useful features used for this paper:

- For a given function $a(x)$ its Fourier transform $\mathcal{F}(a)$ is defined as

$$\mathcal{F}(a)(k) := A(k) := \int_{-\infty}^{\infty} a(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}x \quad (\mathrm{i}^2 = -1), \tag{9}$$

- The derivative $\frac{\partial}{\partial x}$ of a function $a(x)$ can be computed with:

$$\mathcal{F}(\frac{\partial a}{\partial x})(k) = 2\pi \mathrm{i} k A(k) \ . \tag{10}$$

- If $a(x)$ and $b(x)$ are two functions, the Fourier transform of the product of $a(x)$ and $b(x)$ is given by a convolution:

$$\mathcal{F}(a \cdot b)(k) = A(k) * B(k) = \int_{k'=-\infty}^{\infty} A(k-k')B(k')\mathrm{d}k' \,. \tag{11}$$

- A very useful theorem for our model was stated by Plancherel: If we have two functions $a(x)$ and $b(x)$ with Fourier transform $A(k)$ and $B(k)$, we get

$$\int_{-\infty}^{\infty} a(x)\bar{b}(x)\mathrm{d}x = \int_{-\infty}^{\infty} A(k)\bar{B}(k)\mathrm{d}k \,. \tag{12}$$

  with $\bar{b}(x)$ being the conjugate complex of $b(x)$ and $\bar{B}(k)$ the conjugate complex of $B(k)$.

3. Expansion of the deformations u and w

To deal with the derivatives with respect to x and t in an efficient and numerically stable way, the unknown deformations $u(x,z,t,\theta)$ and $w(x,z,t,\theta)$ are Fourier transformed from the space-time domain (x,t) into the wavenumber frequency domain (k,ω). The randomness in the deformations will be handled using a Polynomial Chaos expansion, and in the z-direction we use a finite element ansatz.

The z-range is divided into n_2 equidistant subintervals $[z_j, z_{j+1}]$ ($z_0 = 0$) on which we expand the already Fourier transformed functions $U(k,z,\omega,\theta)$ and $W(k,z,\omega,\theta)$ with simple linear basis functions $g_j(z)$. Together with the Polynomial Chaos expansion the deformations U and W can be expressed as:

$$U(k,z,\omega,\theta) = \sum_{i=0}^{n_1}\sum_{j=1}^{n_2} U_{ij}(k,\omega)g_j(z)\Gamma_i(\boldsymbol{\xi}), \tag{13}$$

$$W(k,z,\omega,\theta) = \sum_{i=0}^{n_1}\sum_{j=1}^{n_2} W_{ij}(k,\omega)g_j(z)\Gamma_i(\boldsymbol{\xi}). \tag{14}$$

The functions $g_j(z)$ are defined as 2^{nd} order B-splines and therefore form a basis:

$$g_0(z) = \begin{cases} 0 & z < z_0 \\ \frac{z_1 - z}{z_1} & z_0 \le z < z_1 \\ 0 & z \ge z_1 \end{cases}, \; g_j(z) = \begin{cases} 0 & 0 \le z < z_{j-1} \\ \frac{z - z_{j-1}}{z_j - z_{j-1}} & z_{j-1} \le z < z_j \\ \frac{z_{j+1} - z}{z_{j+1} - z_j} & z_j \le z < z_{j+1} \\ 0 & z \ge z_{j+1} \end{cases} \text{ for } j > 0; \tag{15}$$

an example for such a basis function is shown in figure 1. As a last step U_{ij} and W_{ij} are discretized over an equidistant grid $(k_\ell, \omega_m) \in [-K, K] \times [-\Omega, \Omega]$, $\ell = 1, \ldots, n_3$, $m = 1, \ldots, n_4$ and the integrals over k and ω are numerically approximated with simple sums[c].

[c]Please note, that this discretization implies a periodic continuation of these functions in the original domain.

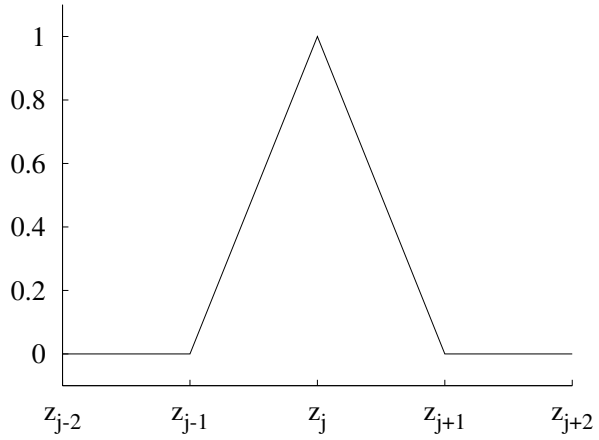

Fig. 1. Basis function $g_j(z)$.

Variation with respect to the the unknowns $\hat{u}_{ij\ell m} := U_{ij}(k_\ell, \omega_m)$ and $\hat{w}_{ij\ell m} := W_{ij}(k_\ell, \omega_m)$ leads to a system of equations

$$\boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}_{\text{Ext}} \ , \tag{16}$$

with stiffness matrix $\boldsymbol{K}$, a vector of external forces $\boldsymbol{f}_{\text{Ext}}$ and the vector $\boldsymbol{x}$ containing the unknowns $\hat{u}_{ij\ell m}$ and $\hat{w}_{ij\ell m}$. To solve Eq. (16), a great deal of computational effort is needed because of the dimension of the matrix involved. To handle this large-scale system efficiently we choose to split it into a (deterministic) mean value part and a stochastic part, analog as to the splitting of the shear modulus. This makes it possible to use the very sparse block diagonal structure of the deterministic part to decouple the system into smaller parts.

4. The mean value part

4.1. *Fourier Transformation*

As already noticed, is the Fourier transformation important for the reduction of the computational effort. To achieve this reduction it is essential to make the Fourier transformation of the system *before* any expansion and variation is made.

Because of the weak formulation and the fact that only products of two functions have to be Fourier transformed, a special case of Plancherel's theorem can be used.

$$\int_{-\infty}^{\infty} a(x)b(x)\mathrm{d}x = \int_{-\infty}^{\infty} A(k)B(-k)\mathrm{d}k \ . \tag{17}$$

Compared with the convolution, Eq. (17) has the advantage that there is no coupling between different wavenumbers in each single product. To illustrate this we arbitrarily

pick the first product $\frac{1-\nu}{1-2\nu}G_0(z)u_x(x,z,t,\theta)^2$ out of L_0 in Eq. (1) and apply a Fourier transformation with respect to x and t.

With a simplified notation neglecting the integral with respect to z and the "constant" factors for the moment, we get:

$$\int_t\int_x u_x(x,z,t,\theta)u_x(x,z,t,\theta)\mathrm{d}x\mathrm{d}t = \int_\omega\int_k 4\pi^2k^2U(k,\omega,z,\theta)U(-k,-\omega,z,\theta)\mathrm{d}k\mathrm{d}\omega\ . \tag{18}$$

Descretization of the above equation (and therefore replacing the integrals with sums) yields[d]

$$c\sum_m\sum_\ell U(k_\ell,z,\omega_m,\theta)U(-k_\ell,z,-\omega_m,\theta). \tag{19}$$

Variation with respect to the unknowns $U(k_\ell,z,\omega_m,\theta)$ shows that only terms containing $U(-k_\ell,z,-\omega_m,\theta)$ remain, since all other elements of the sums become 0. Combined with the orthogonality of the Chaos polynomials (see Sec. 4.2) and the finite element basis functions this results in a 4×4 diagonal block in the system matrix for each finite element and therefore in a blocktridiagonal system matrix K_0.

4.2. *Chaos Polynomial expansion*

For better readability we will use a simpler formulation of the expansions (13) and (14):

$$U(k,z,\omega,\theta)=\sum_i U_i(k,z,\omega)\Gamma_i(\boldsymbol{\xi}),\quad W(k,z,\omega,\theta)=\sum_i W_i(k,z,\omega)\Gamma_i(\boldsymbol{\xi})\ . \tag{20}$$

The dimension of the stochastic space is equal to the number of coefficients used in the KLE ; $\boldsymbol{\xi}:=(\xi_1(\theta),\ldots,\xi_N(\theta))$ is assumed to be a vector of independent random variables $\xi_i(\theta)$ with Gaussian distribution. The orthogonality of the Hermite polynomials Γ_i makes it possible to treat the L_0 part of the system independently for every single Γ_i after calculating the expectation value. For an illustration we pick the first product of two functions out of Eq. (1) and using a simplified notation have:

$$4\pi^2k^2\frac{1-\nu}{1-2\nu}G_0(z)\langle U,U\rangle = 4\pi^2k^2\frac{1-\nu}{1-2\nu}G_0(z)\langle\sum_i U_i\Gamma_i,\sum_j U_j\Gamma_j\rangle = 4\pi^2k^2\frac{1-\nu}{1-2\nu}G_0(z)\sum_i\sum_j U_iU_j\langle\Gamma_i,\Gamma_j\rangle = 4\pi^2k^2\frac{1-\nu}{1-2\nu}G_0(z)\sum_i U_iU_i\mathbb{E}(\Gamma_i^2) \tag{21}$$

with $\mathbb{E}(\Gamma_i^2)=\langle\Gamma_i,\Gamma_i\rangle$ being the expectation value of Γ_i^2. So the deterministic part can be decoupled into smalles subsystems for every single Chaos polynomial.

[d]For better readability we put all constant terms into one constant c.

5. The stochastic part

With the KLE additional functions dependent on x and a vector of random functions $\boldsymbol{\xi}$ were added to the stochastic part L_s of the operator L in Eq. (1). Therefore it is not possible to use either Plancherel's theorem for the Fourier transformation as in Sec. 4, nor the orhogonality properties of the Γ_i when calculating the expectation value. We shall illustrate this with the same example as above, but now in the stochastic version:

$$G_s \sum_i \int_z \int_t \int_x \sqrt{\lambda_i} f_i(x) \xi_i(\theta) u_x(x,z,t,\theta) u_x(x,z,t,\theta) \mathrm{d}x \mathrm{d}t \mathrm{d}z \; . \tag{22}$$

5.1. *Fourier transformation*

Now products of three functions (two deformations and one factor from the KLE) have to be considered. A combination of the convolution and Eq. (17) is used here:

$$\begin{aligned} \int_{x=-\infty}^{\infty} a(x)b(x)c(x)\mathrm{d}x = \int_{x=-\infty}^{\infty} d(x)c(x)\mathrm{d}x = \int_{k=-\infty}^{\infty} D(k)C(-k)\mathrm{d}k = \\ = \int_{k=-\infty}^{\infty} \int_{k'=-\infty}^{\infty} A(k-k')B(k')C(-k)\mathrm{d}k'\mathrm{d}k \; . \end{aligned} \tag{23}$$

For our example Eq. (22) will be transformed into

$$4\pi^2 \sum_i \sqrt{\lambda_i}\xi_i \int_z \int_t \int_k \int_{k'} k k' F_i(k-k') U(k',z,t,\theta) U(-k,z,t,\theta) \mathrm{d}k' \mathrm{d}k \mathrm{d}t \mathrm{d}z \; . \tag{24}$$

For the Fourier transformation over t we can apply Plancherel's theorem as in the mean value case.

Compared to the mean value part the additional integration over k' now results in full coupling between every wavenumber in each single product, because of the additional integration/summation with respect to k'. So instead of a block tridiagonal matrix, the system matrix for this part is dense. In fact the matrix itself will never be constructed explicitly, because just the product of this matrix with a vector are of interest, which can be calculated "on the fly".

5.2. *Chaos Polynomials*

When calculating the expectation value we now must also consider the random functions ξ_i coming from the KLE. With a simplified notation and after omitting all constant factors and the integrals for better readability, we get for Eq. (22)

$$\sum_i F_i(k-k') \mathbb{E}(\xi_i(\theta) U(k',z,\omega,\theta) U(-k,z,-\omega,\theta)) =$$

$$\sum_i F_i(k-k') \mathbb{E}(\xi_i \sum_{j_1} U_{j_1}(k',z,\omega)\Gamma_{j_1}(\boldsymbol{\xi}) \sum_{j_2} U_{j_2}(-k,z,-\omega)\Gamma_{j_2}(\boldsymbol{\xi}) =$$

$$\sum_i \sum_{j_1} \sum_{j_2} F_i(k-k') U_{j_1}(k', z, \omega) U_{j_2}(-k, z, -\omega) \mathbb{E}(\xi_i \Gamma_{j_1} \Gamma_{j_2}) \; . \tag{25}$$

The mixed expectation values $\mathbb{E}(\xi_i \Gamma_{j_1} \Gamma_{j_2})$ are in general non zero, so full coupling between the deformations for all Γ_i is given.

6. Neumann-like iteration

As shown in Sec. 4 it is possible to decouple the mean value part for every Γ_i and every wavenumber k. So $\mathbf{K}_0$ has block diagonal structure: for every Γ_i there is one block $\boldsymbol{K}_{\Gamma_i}$,

$$\boldsymbol{K}_0 = \begin{pmatrix} \boldsymbol{K}_{\Gamma_0} & & \\ & \ddots & \\ & & \boldsymbol{K}_{\Gamma_{n_1}} \end{pmatrix} . \tag{26}$$

All blocks $\boldsymbol{K}_{\Gamma_i}$ are equal except for the constant $\mathbb{E}(\Gamma_i^2)$ and have again block diagonal structure with $n_3 n_4$ subblocks $\boldsymbol{K}_{i,j}, i = 1, \ldots, n_3, j = 1, \ldots, n_4$ (one block for each wavenumber k_i and each frequency ω_j).

Each single block $\boldsymbol{K}_{i,j}$ is built up by 4×4 dimensional matrices, one for each finite element. Due to the simplicity of the finite element basis functions each element is only coupled to at most two other elements (deformations on top of one layer are coupled to the deformations at the bottom of the layer above). Therefore each of the blocks $\boldsymbol{K}_{i,j}$ has a block tridiagonal structure, with subblocks of size 2×2.

On the other hand, it is easy to see that for the stochastic part $\boldsymbol{K}_s$ such a block structure cannot be achieved because there is full coupling between all components as shown in Sec. 5. The idea now is to draw the stochastic part onto the right hand side of the system, which is then solved using the iteration

$$\begin{aligned} \boldsymbol{K}_0 \boldsymbol{x}_0 &= \boldsymbol{f}_{Ext} \\ \boldsymbol{K}_0 \boldsymbol{x}_n &= \boldsymbol{f}_{Ext} - \boldsymbol{K}_s \boldsymbol{x}_{n-1} \; . \end{aligned} \tag{27}$$

We can use the block structure of $\boldsymbol{K}_0$ to solve Eq. (27) efficiently by splitting it into $(n_1+1) n_3 n_4$ subsystems with system matrices $\boldsymbol{K}_{i,j}$ of dimension $2n_2$ (n_1 is the number of Chaos polynomials, n_2 the number of finite elements, n_3 the number of different wavenumbers and n_4 is the number of different frequencies). In general, sparse systems could be solved using a solver based on iteration methods (for example GMRES), but in this case it is more advisable to make some kind of a decomposition (for example a LU-factorization) of the block-tridiagonal matrices because with the Neumann-like iteration the right hand side changes often. There are methods which use the block tridiagonal structure of the linear system to solve it efficiently (for example the URV-decomposition introduced by Gansterer[11]).

The speed of convergence is mainly dependent on the ration $\frac{G_s}{G_0}$. This can be seen by reformulating Eq. (27):

$$x_n = x_0 - S x_{n-1} = x_0 - S(x_0 - S x_{n-2}) = \ldots = \sum_i (-S)^i x_0 \; , \tag{28}$$

where $S := K_0^{-1}K_s$. A sufficient condition for convergence of the iteration is given if $||S||_{Op} < 1$. From finite element theory it is known that K_0 is invertible as long as we stay away from certain singularities which can be dealt with by adding a complex damping factor to the stiffness matrix. K_s has finite dimension and is therefore bounded. The most influential part for $K_0^{-1}K_s$ is the product of G_0^{-1} and the variance factor G_s, which is smaller then G_0 for reasonable cases.

The influence of $\frac{G_s}{G_0}$ can be generally observed in numerical experiments, we shall illustrate that in the following example: We use a model with a layer simulating a fluid, a stochastic layer with 20 finite elements and a halfspace layer at the bottom to prevent unwanted reflections. The main parameters for these layers are summarized in Tab. 1. Three testruns are made, setting the ratio $\frac{G_s}{G_0} = 1/2, 1/20$ and $1/200$ respectively. As a measure

Layer	comp. modul; shear mod.	ρ	ν	d
Fluid	K=2.25E9+2.25E6i	1000		1
Stoch.	G_0=2.00E8+2.00E7i	1800	0.3	20×0.2
Halfsp.	G_0=2.00E8+1.00E7i	1800	0.3	

Table 1. Materialparameters for the different test runs: G_0 is the meanvalue of the shear modulus in N/m^2, ρ the density kg/m^3, ν the Poisson ratio and d the thickness in m, K denotes the compression modulus for the fluid layer in N/m^2.

for the convergence we take the absolute value of the residuum $r := (\boldsymbol{K}_0 + \boldsymbol{K}_s)\boldsymbol{x}_i - \boldsymbol{f}_{\text{Ext}}$. Table 2 shows the residdum for all three runs after each iteration. As it can be seen, the difference between the rows is only in the exponent of the residuum (except for round off errors), mirroring the direct influence of the ratio $\frac{G_s}{G_0}$.

Iter	Run 1	Run 2	Run 3
1	8.263E-08	8.263E-07	8.263E-06
2	2.842E-11	2.842E-09	2.842E-07
3	1.679E-14	1.679E-11	1.679E-08
4	9.848E-18	9.848E-14	9.848E-10
5	7.127E-21	7.027E-16	7.027E-11
6	3.900E-24	3.860E-18	3.860E-12
7	1.326E-23	2.580E-20	2.449E-13

Table 2. Residuum after each iteration for the different ratios G_s/G_0. Run 1: $G_s/G_0 = 1/200$, run 2: $G_s/G_0 = 1/20$, run 3: $G_s/G_0 = 1/2$.

7. Example

As an example we present a model with 20 finite element layers. As external excitation we use a rectangular force of strength 1 N, width 0.4 m and an initial frequency of 100 Hz. The region of interest in the x direction is restricted to the interval [-20m,20m]. For each element we have: a shear modulus G with mean value $G_0 = 200 \cdot 10^6 + 200 \cdot 10^5 \mathrm{iN/m^2}$ and a variance factor of $100 \cdot 10^6 + 100 \cdot 10^5 \mathrm{i\ N/m^2}$, a Poisson ratio of 0.3, a density of ρ=1800 kg/m^3 and a width of 0.2 m. To prevent unwanted reflections at the bottom we add a halfspace layer with boundary condition $u = w = 0$ for $z \to \infty$ (these kinds of layers can be modeled using the theory of elastic waves, see, e.g., Ziegler[25]).

The maximum order of Hermite polynomials being used was set to 4, the number of coefficients in the KLE was 2 (the two highest eigenvalues were chosen) and the wavenumbers k_ℓ ranged from $[-16, 16]$. As covariance function we take $G_s C(x, y) = (100 \cdot 10^6 + 100 \cdot 10^5 \mathrm{i}) \mathrm{e}^{-5|x-y|}$, which will be approximated in the interval $[-20, 20] \times [-20, 20]$ with gridsize $h = 0.02$.

Fig. 2 shows the absolute value of the mean and the standard deviation for the deformation in the x-direction in the first column and the same data for the deformation in the z-direction in the second column.

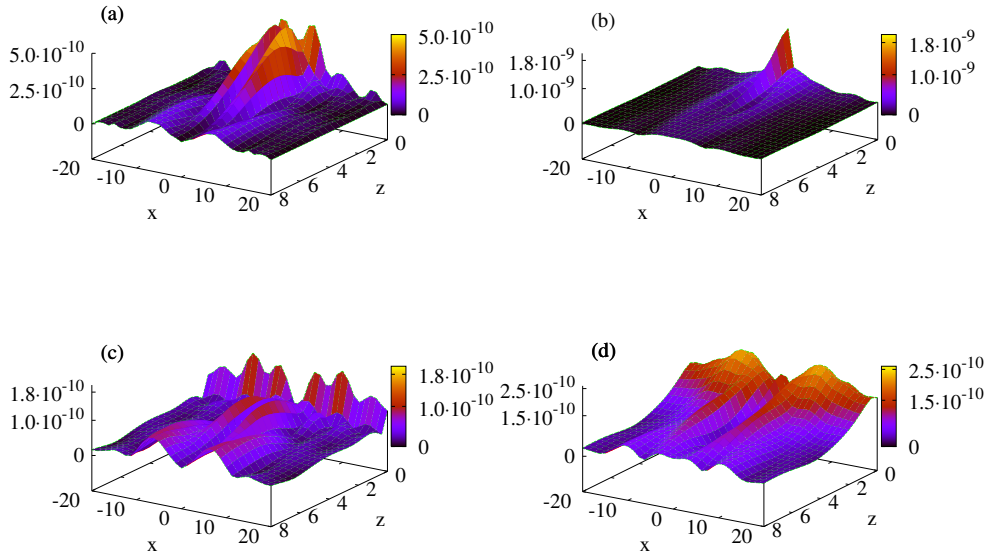

Fig. 2. (a) Absolute of the mean value for the deformation in x direction, (b) absolute of the mean value for the deformation in z direction, (c) standard deviation for the deformation in x direction, (d) standard deviation for the deformation in z direction. All plots show the data in the frequency domain at $\omega = 100$Hz.

8. Conclusion and Outlook

A 2-D model for simulating vibrations in ground layers with stochastic shear modulus was presented. The shear modulus was split up into a deterministic and a stochastic part using the Karhunen-Loewe expansion, which also made it possible to split the whole system in the same manner. The unknown deformations, which are dependent on the stochastic parameter θ, were expanded using finite elements and Polynomial Chaos expansion. To keep the dimensions of the matrices involved in the numerical solution of the system small, a Neumann-like iteration was used. Finally results for a simple example demonstrated the feasibility of the approach.

Future work will include expanding this model into three dimension and implementing a moving external force. Wavelet compression will be considered for reconstruction of the deformations in the time-space domain, as presented by Lieb[16]. To get more flexibility for the finite element method, other spline functions can be used. Finally it should be investigated, whether modern results of frame theory[10] can give new insights to this problem. In recent work Balazs[4,5] gives some suggestions for a connecting frame theory, reconstruction and numerical efficient algorithms.

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