Mathematics in Acoustics

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Overview:

1. Applied Mathematics

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2. Numerical Mathematics

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2. Numerical Mathematics
3. Application-oriented Mathematics

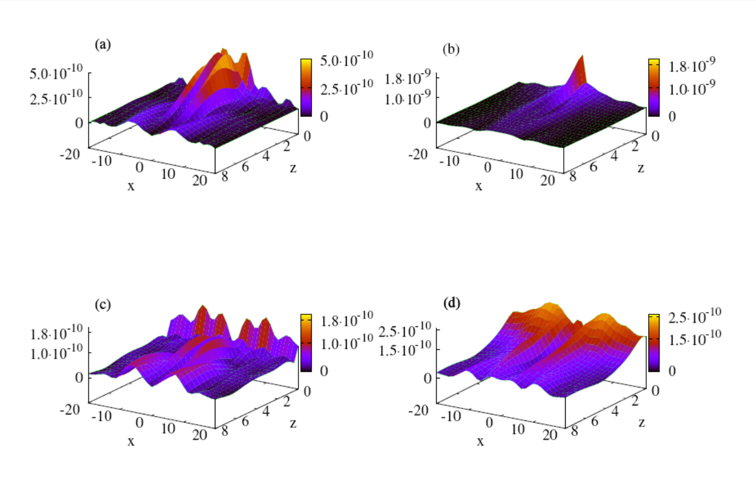
Applied Mathematics, part 1

$$\delta \int_t \int_x L \, \mathrm{d}x \mathrm{d}t = 0 \, ,$$

$$L = \int_z \frac{1}{2} \left[\left(\frac{2(\nu+1)}{1-2\nu} G(x,z,\theta) \right) (u_x^2 + w_z^2) + \frac{4\nu}{1-2\nu} G(x,z,\theta) u_x w_z + \right.$$

$$\left. + \; G(x,z,\theta)(u_z + w_x)^2 \right] - \frac{1}{2}\rho(u_t^2 + w_t^2) dz - f_{\mathrm{Ext}} w_{|z=0} \, . \quad (1)$$

Applied Mathematics: Vibrations (Numerical Acoustics)



3 mathematicians

Signal Processing : Time Frequency Analysis

Short Time Fourier Transformation (STFT)

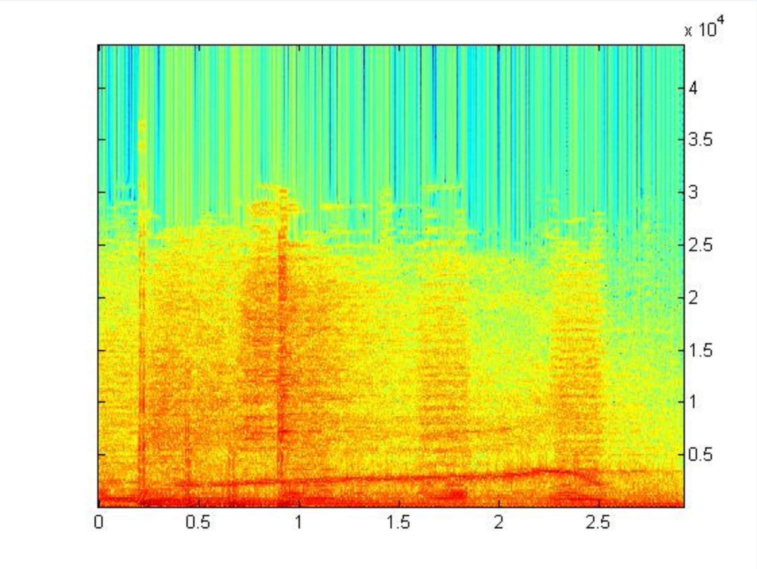
Definition

Let $f, g \neq 0$ in $L^2\left(\mathbb{R}^d\right)$, then we call

$$\mathcal{V}_g f(\tau, \omega) = \int_{\mathbb{R}^d} f(x) \overline{g(x-\tau)} e^{-2\pi i \omega x} dx .$$

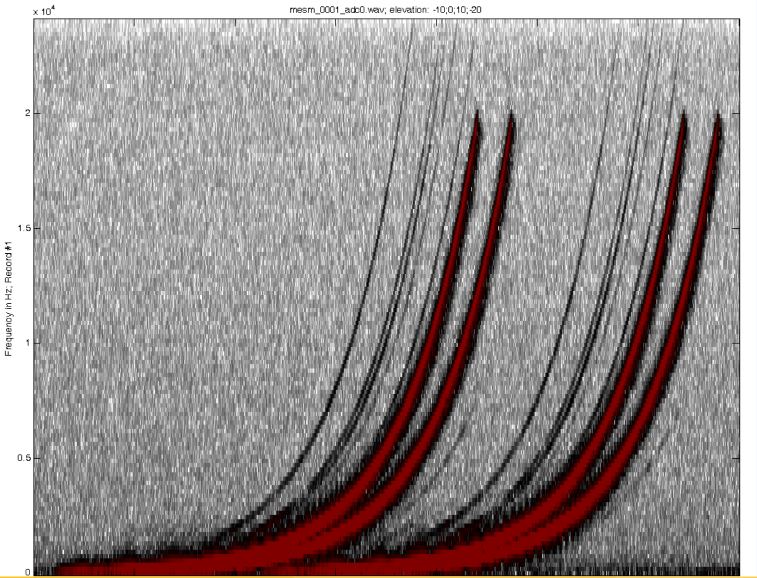
the Short Time Fourier Transformation (STFT) of the signal f with the *window* g.

Short Time Fourier Transformation (STFT)



Applied Mathematics, part 2

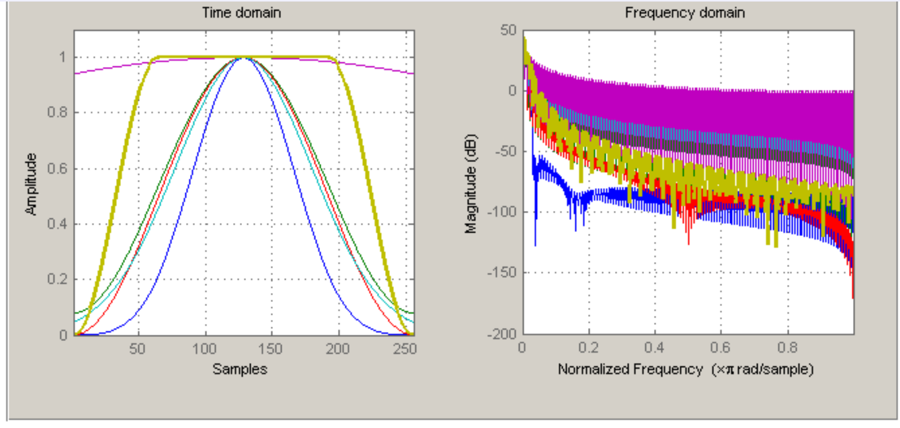
Multiple Exponentiell Sweeps Method



Numerical Mathematics

Perfect Reconstruction Resynthesis I

Commonly used windows and their spectra:



Perfect Reconstruction Resynthesis II

Overlap Add: Comparison of Errors for $N_{win} = 1024$ and overlap $= 50\%$.:

window \ error	max. rel. error	rel. err. rand. sig.	rel. err. audio sig.
Hanning	0.00153547	0.000471961	0.000468024
Hamming	0.0013077	0.000399719	0.00039863
Rectangular	0	$3.02468e-008$	$1.69078e-008$
Bartlett	0	$6.49745e-008$	$3.72214e-008$
Blackman Harris	1.30664	0.268622	0.26723
Trunc. Gaussian	0.148915	0.0503265	0.0499618
Kaiser ($\beta = 0.5$)	0.0151726	0.00462928	0.00444201
Tukeywin	0.999979	0.227209	0.2303

Perfect Reconstruction Resynthesis III

Frame theory $\Longrightarrow$ perfect reconstruction

window \ error	rel. err. audio sig. (50%)	rel. err. audio sig. (25%)	rel. err. audio sig. (12.5%)
Hanning	$2.34753e-008$	$1.76303e-008$	$1.24641e-008$
Hamming	$2.30809e-008$	$1.71258e-008$	$1.21008e-008$
Rectangular	$2.092e-016$	$1.63434e-016$	$1.25851e-016$
Bartlett	$2.25018e-008$	$1.69021e-008$	$1.20049e-008$
Blackman Harris	$2.45796e-008$	$2.08936e-008$	$1.49229e-008$
Trunc. Gaussian	$2.33804e-008$	$1.78396e-008$	$1.26315e-008$
Kaiser ($\beta = 0.5$)	$1.80255e-008$	$1.28678e-008$	$9.04746e-009$
Tukeywin	$7.27761e-009$	$6.56274e-009$	$4.81764e-009$

Table: *'dual' method: Comparison of relative Errors for $N_{win} = 1024$ and different overlaps.*

Double Preconditioning I

To find dual window efficiently:

$$P = C\left(D\left(S\right)^{-1}\cdot S\right)^{-1} D(S)^{-1}$$

Figure: The double preconditioning matrix

- Parameter: $g,\ a, b$
- Initialization: $B = block(g, a, b)$
- Preconditioning :

$$P_1 = inv_{block}\left(diag_{block}\left(B\right)\right)$$

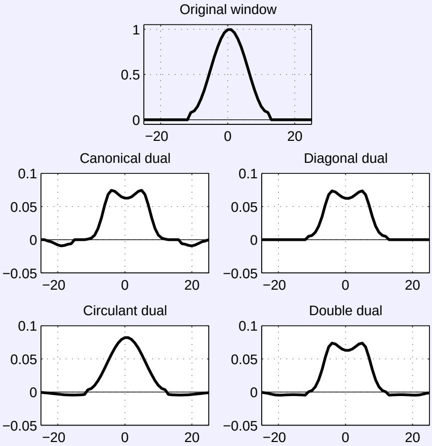
$$S_1 = P_1 \bullet_{block} B$$

$$P_2 = inv_{block}\left(circ_{block}\left(S_1\right)\right)$$

$$S_2 = P_2 \bullet_{block} S_1$$

Figure: The double preconditioning algorithm

Double Preconditioning II



Application-oriented Mathematics

Abstract Nonsense with Motivation in Applications

Frames I : definition

Definition

The sequence $(g_k | k \in K)$ is called a frame for the Hilbert space $\mathcal{H}$, if constants $A, B > 0$ exist, such that

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, g_k \rangle|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \ \forall \ f \in \mathcal{H}$$

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- Gabor frame : $(g_{m,n}) = (M_{nb} T_{ma} g)$ for some a, b.
- frames = "spanning systems in $\mathcal{H}$"
- frames = generalization of bases
- frame condition = generalization of Parseval's theorem
- Perfect reconstruction is guaranteed with the 'canonical dual frame' $\tilde{g}_k = S^{-1} g_k$ with S the frame operator (i.e. combined analysis/resynthesis operator).

Frame Multiplier I : definition

Definition

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert-spaces, let $(g_k)_{k \in K}$ be a frame in $\mathcal{H}_1$, $(f_k)_{k \in K}$ in $\mathcal{H}_2$. Define the operator $\mathbf{M}_{m,(f_k),(g_k)} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, the frame multiplier for these frames as the operator

$$\mathbf{M}_{m,(f_k),(g_k)} f = \sum_k m_k \langle f, g_k \rangle f_k$$

where $m \in l^\infty(K)$ is called the *symbol*.

Frame Multiplier II : exemplary new theoretical result

Theorem

Let $\mathbf{M}_{m,f_k,g_k}$ be a frame multiplier for $\{g_k\}$ and $\{f_k\}$ with the upper frame bounds B and B' respectively. Then

1. *If $m \in l^\infty$ $\mathbf{M}$ is a well defined bounded operator.* $\|\mathbf{M}\|_{Op} \leq \sqrt{B'}\sqrt{B} \cdot \|m\|_\infty$.
2. $\mathbf{M}^*_{m,f_k,g_k} = \mathbf{M}_{\overline{m},g_k,f_k}$. *Therefore if m is real-valued and $f_k = g_k$, $\mathbf{M}$ is self-adjoint.*
3. *If $m \in c_0$, $\mathbf{M}$ is compact.*
4. *If $m \in l^1$, $\mathbf{M}$ is a trace class operator with* $\|\mathbf{M}\|_{trace} \leq \sqrt{B'}\sqrt{B}\ \|m\|_1$. *And* $tr(M) = \sum_k m_k \langle f_k, g_k \rangle$.
5. *If $m \in l^2$, $\mathbf{M}$ is a Hilbert Schmidt operator with* $\|\mathbf{M}\|_{\mathcal{HS}} \leq \sqrt{B'}\sqrt{B}\ \|m\|_2$.

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